AN ALGEBRAIC FORMULATION OF ASYMPTOTICALLY SEPARABLE QUANTUM MECHANICS

Derek McLean

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by

R. G. Derek McLean.

St. Leonard's College

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DECLARATIONS

I R.G. Derek McLean hereby certify that this thesis which is approximately 26,000 words in length has been written by me, that it is the record of work carried out by me, and that it has not been submitted in any previous application for a higher degree.

13th December 1983 D.McLean

I hereby certify that the candidate has fulfilled the conditions of the Resolution and regulations appropriate to the degree of Doctor of Philosophy in the University of St. Andrews and that he is qualified to submit this thesis in application for that degree.

13th December 1983 K.K.Wan

Research Supervisor

DECLARATION

I was admitted as a research student under Ordinance No. 12 on 1st October 1979, and as a candidate for the degree of Ph.D. on 1st October 1980; the higher study for which this is a record was carried out in the University of St. Andrews between 1979 and 1983.

13th December 1983 D. McLean

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ABSTRACT.

This thesis explores the possibility of an algebraic formulation of non-relativistic quantum theory in which certain paradoxes associated with non-locality may be resolved.

It is shown that the localisation of a free quantum mechanical wave function at large time coincides approximately with the localisation of an ensemble of classical particles having the same momentum range. This result is used to give a formal definition of spatially separating states and spatially separating particles.

We then study certain C*-algebras on which expectation values converge in an infinite time limit. By considering such algebras which contain local observables it is possible to introduce states at infinity as limits of states described by wave functions. In such a state at infinity there is zero probability of a position measurement finding the system in any bounded region in configuration space.

It is shown that a C*-algebra exists on which any coherent superposition of spatially separating states will converge in an infinite time limit to a mixture of disjoint states. This allows us to obtain an asymptotic resolution of de Broglie's paradox and the Einstein, Podolsy and Rosen paradox.

These results are obtained for the simplest types of quantum systems i.e. a one particle system without spin having configuration space $\mathbb{R}^n$ and a system consisting of two such particles which may be distinguished from each other.

CONTENTS.

CHAPTER V. A FURTHER STUDY OF LOCAL OBSERVABLES.

0. INTRODUCTION.

The problem of nonlocality in quantum mechanics has attracted much attention over the years. This thesis contains a discussion of this problem within the C*-algebra approach to quantum mechanics. The possibility of an algebraic formulation of quantum mechanics which incorporates nonlocality only over small distances and which is separable at large distances is investigated. Our results are applied to tackle the de Broglie paradox and the Einstein, Podolsky and Rosen paradox for spinless particles.

Chapter I contains some short reviews of the relevant background material emphasising those aspects which will appear in the main body of the work.

In chapter II we show that the localisation of a free wave function at large times coincides approximately with the localisation of a classical particle having the same momentum range. This result is then extended to scattering states when the system is no longer free. A precise definition of spatially separating states and spatially separating systems is then formulated.

Chapter III begins with a study of the C*-algebra generated by the local observables defined by Wan and Jackson (1983) in a preprint entitled "On the Localization of Observables in Quantum Mechanics I: Bounded Observables". We then introduce the notion of an asymptotic algebra in which certain expectation values converge in time. It is shown that there are nontrivial examples of such algebras and the properties of certain asymptotic algebras containing local observables are investigated.

In chapter IV the results of chapters II and III are combined in a discussion of spatially separating states as states on a particular operator algebra. The notions of states and observables at infinity are introduced and it is shown that a superposition of two coherent spatially separating states can evolve asymptotically into a mixture of disjoint states. This enables us to effectively tackle the quantum-mechanical paradoxes associated with nonlocality.

In chapter V we introduce a generalisation of the local observables of chapter III. We define localisation with respect to a general spectral measure and then study the properties of our local observables and the algebra they generate. This chapter concludes with a discussion of the lattice structure of the projections in the algebra of generalised local observables.

Section 6 contains a shorter and neater proof of results contained in papers by Wan and myself (Physics Letters 94A (1983) 198, 95A (1983) 76). Two papers (also by Wan and myself) entitled "Asymptotic Operator Algebras in Quantum Mechanics" and "Observables of Asymptotically Vanishing Correlations, States at Infinity and Quantum Separability" which are based on some of the contents of sections 8,9 and 10 have recently been accepted for publication by Journal of Physics A.

A reference of the form (m.n) refers to section m, subsection n while one of the form (An) refers to section n in the appendix.

We conclude this section with a brief summary of notation and terminology.

The symbols $\subseteq$, $\subset$ and $-$ denote the set theoretic relations of inclusion, proper inclusion and relative complement respectively. The symbols $\mathbb{R}$, $\mathbb{C}$ and $\mathbb{N}$ denote respectively the real and complex numbers and the set $\{1,2,3,\ldots\ldots\}$. If S is a set then χ_S will denote the characteristic function of S defined by

$$\chi_S(x) = \begin{cases} 1 & (x \in S) \\ 0 & (x \notin S) \end{cases}$$

A subset of $\mathbb{R}^n$ will be called an interval if it is a Cartesian product of intervals of the real line.

If R and S are subsets of a vector space V then $R + S$ will denote the set $\{z \in V : z = x + y$ for some $x \in R$ and some $y \in S\}$ and if a is a scalar then we shall write aS to denote the set $\{z \in V : z = ax$ for some $x \in S\}$. A non-empty subset of a vector space will be called a linear manifold if it contains all linear combinations of its elements. A closed linear manifold in a Hilbert Space will be called a subspace.

The domain and range of a linear operator A will be denoted by $\mathrm{dom}(A)$ and $\mathrm{ran}(A)$ respectively. A self-adjoint operator in a Hilbert space $\mathcal{H}$ is a densely defined operator which is equal to its adjoint. The set of all bounded linear operators on $\mathcal{H}$ will be denoted by $\mathbb{B}(\mathcal{H})$. An element of $\mathbb{B}(\mathcal{H})$ (or more generally of any C*-algebra) will be called Hermitian if it is equal to its adjoint. If M is a projection on some Hilbert space then $M^{\perp}$ will denote the orthogonal complement of M. If X is an element of a C*-algebra $\mathcal{A}$ then we denote by $X\mathcal{A}X$ and $\mathbb{C}X$ the sets $\{A \in \mathcal{A} : A = XBX$ for some $B \in \mathcal{A}\}$ and $\{A \in \mathcal{A} : A = aX$ for some $a \in \mathbb{C}\}$ respectively. The (multiplicative) identity in a C*-algebra will be denoted by I.

CHAPTER I

REVIEW OF BACKGROUND MATERIAL.

1. Hilbert Space Quantum Mechanics.
2. Quantum Mechanics in $L^2(\mathbb{R}^n)$.
3. Quantum Mechanics and Non-locality.
4. Algebraic Quantum Mechanics.
5. Geometric Scattering Theory.

1. HILBERT SPACE QUANTUM MECHANICS.

In an axiomatic formulation of quantum mechanics it is usually assumed that the states and observables of a physical system can be described by suitable mathematical objects. In this section we review the Hilbert space formulation of elementary (non-relativistic) quantum theory from an axiomatic viewpoint. In this approach the mathematical objects are Hilbert space operators and the relationship with the basic physical concepts is given by the following axiom,

(1.1) States and Observables.

To each quantum mechanical system there corresponds a Hilbert space. If $\mathcal{H}$ is the Hilbert space describing such a system then the observables are in one-to-one correspondence with the (not necessarily bounded) self-adjoint operators in $\mathcal{H}$. The states are in one-to-one correspondence with the density operators (i.e. positive trace class operators having unit trace) on $\mathcal{H}$.

We also have an axiom which allows the theory to make predictions,

(1.2) The Probabilistic Interpretation.

If A is a self-adjoint operator in $\mathcal{H}$ and E_A is the spectral measure of A then for each Borel set S of $\mathbb{R}$ the number $\mathrm{Tr}(WE_A(S))$ represents the probability that a measurement of the observable represented by A will give a value in S when the system is in the state represented by the density operator W.

For such an A and W a probability measure $p_{A,W}$ may be defined on the Borel sets of $\mathbb{R}$ by the formula $p_{A,W}(S) = \mathrm{Tr}(WE_A(S))$. If A is bounded then the expectation value of A in the state W is defined to be $\int x dp_{A,W}(x) = \mathrm{Tr}(WA)$.

To describe the time evolution of the system it will be useful to introduce the notion of an evolution group. An evolution group in a Hilbert space $\mathcal{H}$ is defined to be a continuous homomorphism from the group $\mathbb{R}$ to the group of unitary operators on $\mathcal{H}$ where the latter is equipped with the strong operator topology. We shall confine our attention to the "Schrodinger picture" in which the states change in time.

(1.3) Time Evolution.

The time evolution of the system is represented by an evolution group in the appropriate Hilbert space. If the initial state is given by a density operator W then the state at time t is given by the density operator $U_t^* W U_t$ where U is the evolution group.

A state is said to be pure if the corresponding density operator W is a one-dimensional projection. In this case any unit vector f in the range of W is called a state vector of the system. For such a state we have

$$\mathrm{Tr}(WE_A(S)) = \|E_A(S)f\|^2 \quad \text{and} \quad \mathrm{Tr}(WA) = \langle f | Af \rangle,$$

where E_A is the spectral measure of the self-adjoint operator A.

A state which is not pure is called mixed. If U is an evolution group then by Stone's theorem there is a unique self-adjoint operator H (called the Hamiltonian of the system) such that $U_t = \exp(Ht/i\hbar)$ for all $t \in \mathbb{R}$. Here $\hbar$ is a positive real number representing Planck's constant.

An initial pure state described by a state vector f will remain pure at any later time t and will have state vector $U_t f$. When f is in the domain of the Hamiltonian H then $U_t f$ is given by Schrödinger's equation

$$i\hbar \frac{d}{dt} (U_t f) = H(U_t f).$$

The interaction of a quantum mechanics system with a measuring instrument cannot in general be characterised by an evolution group in the Hilbert space associated with the system. For certain measurements (often called measurements of the first kind) which give the same outcome upon immediate repetition the change of state is described by von Neumann's projection postulate or its generalisation the Lüdders rule.

Throughout this work we shall use the term "measurement" to mean a measurement where the change of state obeys one of these rules. Other types of measurement will not be considered.

(1.4) The Projection Postulate.

Let B be an observable with a pure point spectrum whose eigenvalues all have multiplicity one. If a measurement of B gives an eigenvalue b then the state after measurement is an eigenvector of B corresponding to the eigenvalue b.

(1.5) The Lüdders Rule for Observables with Pure Point Spectra.

Let B be an observable with a pure point spectrum and let E_r ($r \in I$) be the projections on to the eigenspaces of B. If the system is initially in the state W and a measurement of B results in the eigenvalue corresponding to E_r then the state W' immediately after the measurement is given by

$$W' = \frac{E_r W E_r}{\mathrm{Tr}(E_r W E_r)} .$$

In particular when W is pure the initial state vector f and final state vector f' are related by $f' = \frac{1}{\|E_r f\|} E_r f$.

To generalise the Lüdders rule to an arbitrary observable note that since any isolated point of a self-adjoint operator is an eigenvalue (Weidmann (1980) p202) it follows that any point in the continuous spectrum will have a neighbourhood lying in the continuous spectrum. Thus a point in the continuous spectrum cannot be distinguished from nearby points by a measuring device whose "pointer" has a finite width. So in general a measurement will only restrict the value of an observable to a small interval. The change of state during such a measurement is given by the following version of the Lüdders rule.

(1.6) The Lüdders Rule.

If a measurement of an observable A restricts the value of A to a Borel set R and the system was initially in the state W then the final state W' is given by

$$W = \frac{E_A(R)\, W E_A(R)}{\mathrm{Tr}\,(E_A(R) W E_A(R))}$$

where E_A is the spectral measure of A.

The final axiom gives a rule for combining two distinct systems.

(1.7) Combined Systems.

If two individual systems are described by Hilbert spaces $\mathcal{H}_1$ and $\mathcal{H}_2$ then the Hilbert space of the combined system is the tensor product $\mathcal{H}_1 \otimes \mathcal{H}_2$ of $\mathcal{H}_1$ and $\mathcal{H}_2$. Operators of the form $A \otimes I$ where A is a Hermitian operator in $\mathcal{H}_1$ represent bounded observables of the first system and those of the form $I \otimes B$ where B is a Hermitian operator in $\mathcal{H}_2$ represent bounded observables of the second system.

(1.8) General References.

The Hilbert Space formulation of Quantum Mechanics:
von Neumann(1955), Jordan(1969), Prugovečki(1981), Gudder(1979), Packel(1974).

The Projection Postulate and the Lüdders Rule:
von Neumann(1955) [chapter III, section 3], Lüdders(1951), Furry(1966), Jauch(1968) [chapter 11, section 3], Bub(1979a), Bub(1979b).

Combined Systems:
Jauch(1968) [chapter 11, sections 7,8], Bub(1974).

2. QUANTUM MECHANICS IN $L^2(\mathbb{R}^n)$.

For a quantum mechanical particle with configuration space $\mathbb{R}^n$ the appropriate Hilbert space may be realised as $L^2(\mathbb{R}^n)$ for a particle without spin. It will be convenient to define the position and momentum observables as spectral measures on the Borel sets of $\mathbb{R}^n$ rather than as self-adjoint operators in $L^2(\mathbb{R}^n)$ (for example this is a special case of the definition of observable given in Piron (1976)).

(2.1) The position spectral measure E_Q in $L^2(\mathbb{R}^n)$ is defined for all Borel sets S in $\mathbb{R}^n$ by

$$E_Q(S)f = \chi_S f, \qquad (f \in L^2(\mathbb{R}^n)).$$

If $u : \mathbb{R}^n \longrightarrow \mathbb{C}$ is a Borel function then we can define an operator $u(Q)$ in $L^2(\mathbb{R}^n)$ by

$$u(Q) = \int u \, dE_Q .$$

Then $u(Q)$ is the multiplication operator induced by u i.e.

$$dom(u(Q)) = \{ f \in L^2 : uf \in L^2 \},$$

$$u(Q)f = uf \quad \text{for all } f \in dom(u(Q)).$$

If u is real valued then $u(Q)$ is self-adjoint and if $u \in L^\infty(\mathbb{R}^n)$ then $u(Q) \in \mathbb{B}(L^2(\mathbb{R}^n))$. We denote by $L^\infty(Q)$ the set $\{u(Q) : u \in L^\infty(\mathbb{R}^n)\}$. $L^\infty(Q)$ is a von Neumann algebra and $L^\infty(Q)' = L^\infty(Q)$.

(2.2) The Fourier transform in $L^2(\mathbb{R}^n)$ is the operator F defined for $f \in L^1(\mathbb{R}^n) \cap L^2(\mathbb{R}^n)$ by

$$(Ff)(y) = (2\pi\hbar)^{-n/2} \int f(x) \exp(-ixy/\hbar)\, dx \qquad (\forall y \in \mathbb{R}^n)$$

and extended by continuity to $L^2(\mathbb{R}^n)$. Here $\hbar$ is a positive real number representing Planck's constant. The parity operator $\mathcal{P}$ in $L^2(\mathbb{R}^n)$ is defined by

$$(\mathcal{P}f)(x) = f(-x) \qquad (\forall f \in L^2(\mathbb{R}^n))(\forall x \in \mathbb{R}^n).$$

F and $\mathcal{P}$ are both unitary and $F^2 = \mathcal{P}$ so $F^4 = \mathcal{P}^2 = I$.

(2.3) The momentum spectral measure E_P in $L^2(\mathbb{R}^n)$ is defined for all Borel sets S in $\mathbb{R}^n$ by

$$E_P(S) = F^{-1} E_Q(S) F.$$

where E_Q and F are the position spectral measure and Fourier transform defined above. Now for each measurable function $u : \mathbb{R}^n \longrightarrow \mathbb{C}$ we can define an operator u(P) by

$$u(P) = F^{-1} u(Q) F = \int u \, dE_P$$

and since F is unitary u(P) is self-adjoint whenever u is real-valued and $u(P) \in \mathcal{B}(L^2(\mathbb{R}^n))$ whenever $u \in L^\infty(\mathbb{R}^n)$. We denote by $L^\infty(P)$ the set of all operators of the form u(P) where $u \in L^\infty(\mathbb{R}^n)$. It follows that $L^\infty(P)$ is a von Neumann algebra and $L^\infty(P)' = L^\infty(P)$.

(2.4) For each density operator W the mapping $S \longmapsto \mathrm{Tr}(WE_Q(S))$ is a probability measure on the Borel sets of $\mathbb{R}^n$ and we interpret $\mathrm{Tr}(WE_Q(S))$ as the probability that a position measurement when the state is W will find the particle in S. Similarly $\mathrm{Tr}(WE_P(R))$ represents the probability that a momentum measurement gives a value in R.

(2.5) We may now define self-adjoint operators Q^2 and P^2 in $L^2(\mathbb{R}^n)$ by taking $u(x) = x^2$ $(\forall x \in \mathbb{R}^n)$ and letting $Q^2 = \int u \, dE_Q$ and $P^2 = \int u \, dE_P$. The free particle Hamiltonian in $L^2(\mathbb{R}^n)$ for a particle of mass m is defined by $H_o = \frac{1}{2m} P^2$ where m is a positive real constant. Since H_o is self-adjoint the formula $U_t = \exp(-itH/\hbar)$ $(t \in \mathbb{R})$ defines an evolution group U which will be called the free particle evolution group in $L^2(\mathbb{R}^n)$ for a particle of mass m.

(2.6) The combined system consisting of two distinguishable particles each with configuration space $\mathbb{R}^n$ has Hilbert space $L^2(\mathbb{R}^n) \otimes L^2(\mathbb{R}^n)$. There is a unique unitary operator $\Omega : L^2(\mathbb{R}^n) \otimes L^2(\mathbb{R}^n) \longrightarrow L^2(\mathbb{R}^{2n})$ such that

$$(\Omega f \otimes g)(x,y) = f(x)\, g(y) \qquad (\forall f, g \in L^2(\mathbb{R}^n))\ (\forall x, y \in \mathbb{R}).$$

Ω will be called the natural unitary operator between these spaces. If F_n denotes the Fourier transform in $L^2(\mathbb{R}^n)$ for each n then $\Omega (F_n \otimes F_n) \Omega^{-1} = F_{2n}$.

For a system of two non-identical particles each with configuration space $\mathbb{R}^n$ we may therefore identify the Hilbert space of the combined system with $L^2(\mathbb{R}^{2n})$, and regard $f \otimes g$ as the element of this space defined by

$$(f \otimes g)(x,y) = f(x)\, g(y) \qquad (\forall x, y \in \mathbb{R}^n)$$

whenever $f, g \in L^2(\mathbb{R}^n)$. Similarly if $A, B \in \mathbb{B}(L^2(\mathbb{R}^n))$ we identify $A \otimes B$ with the operator defined on "simple tensors" by

$$((A \otimes B)(f \otimes g))(x,y) = ((Af)(x))\,((Bg)(y))$$

and extended to $L^2(\mathbb{R}^{2n})$.

For a fixed n we shall often denote the position and momentum spectral measures in $L^2(\mathbb{R}^n)$ by E_Q and E_P (respectively) and those in $L^2(\mathbb{R}^{2n})$ by $E_{Q\otimes Q}$ and $E_{P\otimes P}$ (respectively). The position of the first particle is described by the spectral measure $E_{Q\otimes I}$ defined on the Borel sets of $\mathbb{R}^n$ by

$$E_{Q\otimes I}(S) = E_Q(S)\otimes I = E_{Q\otimes Q}(S\times\mathbb{R}^n),$$

i.e. $Tr(W(E_{Q\otimes I}(S)))$ is the probability that a measurement of the position of the first particle will find it localised in S when the combined system is in the state W. The position of the second particle is described by the spectral measure $E_{I\otimes Q}$ defined on the Borel sets of $\mathbb{R}^n$ by

$$E_{I\otimes Q}(S) = I\otimes E_Q(S) = E_{Q\otimes Q}(\mathbb{R}^n\times S).$$

Similarly the spectral measures $E_{P\otimes I}$ and $E_{I\otimes P}$ describing the momenta of the first and second particles are given by

$$E_{P\otimes I}(S) = E_P(S)\otimes I = E_{P\otimes P}(S\times\mathbb{R}^n),$$

$$E_{I\otimes P}(S) = I\otimes E_P(S) = E_{P\otimes P}(\mathbb{R}^n\times S)$$

(respectively) where S is a Borel set in $\mathbb{R}^n$.

Finally a free evolution group U in $L^2(\mathbb{R}^{2n})$ is defined by

$$U_t = \exp\left(\tfrac{-it}{2m_1\hbar}P^2\right)\otimes\exp\left(\tfrac{-it}{2m_2\hbar}P^2\right)$$

where m_1 and m_2 are positive real numbers representing the masses of the individual particles. We have $U_t = \exp(tH/i\hbar)$ where the self-adjoint operator H is the closure of the operator $\frac{1}{2m_1}P^2\otimes I + \frac{1}{2m_2}I\otimes P^2$ in $L^2(\mathbb{R}^{2n})$. H is given in terms of the momentum spectral measure $E_{P\otimes P}$ in $L^2(\mathbb{R}^{2n})$ by $H = \int u\, dE_{P\otimes P}$ where $u:\mathbb{R}^{2n}\longrightarrow\mathbb{R}$ is defined by

$$u(y_1,\ldots\ldots y_n) = \frac{1}{2m_1}(y_1^2+\cdots+y_n^2) + \frac{1}{2m_2}(y_{n+1}^2+\cdots+y_{2n}^2)$$

$(y_1,\ldots\ldots y_{2n}\in\mathbb{R})$.

(2.7) An evolution group U in $L^2(\mathbb{R}^n)$ will be called a free evolution group if either U is the free evolution group of a particle of mass m for some m (2.5) or if (when n is even) U is the free evolution group of a system of two particles as defined above. If U is a free particle evolution group in $L^2(\mathbb{R}^n)$ then the parity operator commutes with U_t for every $t \in \mathbb{R}$.

(2.8) General References.

Amrein (1981), Amrein, Jauch and Sinha (1977), Jauch (1968), von Neumann (1955), Prugovečki (1981).

(2.9) Notes.

For a definition of the integral of a measurable function with respect to a spectral measure see Rudin (1973) (pp 341-345); the special case of a bounded function is covered in Halmos (1957) (pp 60-61). The assertion that u(Q) is a multiplication operator is easily proved (e.g. apply exercise 6.7 in chapter III of Prugovečki (1981)). For a proof that $L^\infty(Q)$ is a von Neumann algebra with $L^\infty(Q)' = L^\infty(Q)$ see Dixmier (1969) (p118). A definition of the Fourier transform which involves Planck's constant may be found in Prugovecki (1981) (pp 218-9). The relation $P = F^2$ follows from Weidmann (1980) (pp 291-292).

For further details of the natural unitary operator Ω see Reed and Simon (1972) (pp 49-53). The relation between Ω and the Fourier transforms follows immediately from Fubini's theorem. The relation

$E_Q(S)\otimes I = E_{Q\otimes Q}(S\times\mathbb{R}^n)$ is easily proved and $E_P(S)\otimes I = E_{P\otimes P}(S\times\mathbb{R}^n)$ now follows from the connection between Ω and the Fourier transforms. The formula for the generator of the two particle evolution group comes from Weidmann (1980) (p267) and to prove $H = \int u\, dE_{P\otimes P}$ Fourier transform everything to get a simple equality between multiplication operators. The Fourier transform may also be used to show that the parity operator P commutes with a free evolution group (since $P = F^2$ is invariant under the Fourier transform). Details of the connection between Fourier transforms, multiplication and differential operators may be found in Weidmann (1980) (p299) or Amrein(1981) (pp42-43).

3. QUANTUM MECHANICS AND NON-LOCALITY.

This section contains a brief outline of two quantum mechanical paradoxes which involve the idea of "spatial separation". These paradoxes arise by assuming that the formal structure of quantum mechanics outlined in section 1 is compatible with the following assertion which we shall call Einstein locality.

(3.1) Einstein Locality. If two systems which have interacted in the past are now arbitrarily distant then anything done to one system does not affect the other in any way.
(Selleri and Tarozzi (1981) p7).

(3.2) We first discuss the "Einstein Podolsky and Rosen paradox". The simplified version described here is essentially the same as that given in Jauch(1968).

Assume that two particles are spatially separating and that they cease to interact when sufficiently far apart. Suppose also that when they are far apart the state vector of the combined system is

$$h = a(f_1 \otimes f_2) + b(g_1 \otimes g_2)$$

where f_1 and g_1 are orthogonal unit vectors in the Hilbert space of the first system , f_2 and g_2 are orthogonal unit vectors in the Hilbert space of the second system and $a, b \in \mathbb{C}$ with $|a|^2 + |b|^2 = 1$, $a \neq 0 \neq b$. Let A be a hermitian operator on the Hilbert space of the first system having f_1 and g_1 as eigenvectors corresponding to distinct eigenvalues. A measurement of the observable A of the first system is regarded as a measurement of $A\otimes I$ on the composite system and (by the measurement theory outlined in section 1) when the initial state is h a measurement of $A\otimes I$ will cause the state to change to either $f_1\otimes f_2$ or $g_1\otimes g_2$. Hence after the measurement the state vector of the second system is f_2 or g_2.

Assuming Einstein locality no change in the state of the second system can take place as a result of the measurement performed on the first, and hence the second system must have been in one of the states f_2 or g_2 before the measurement. It now follows that the combined system was in one of the states $f_1\otimes f_2$ or $g_1\otimes g_2$ before the measurement. Now the initial state h as given was pure and we have deduced that the state before the measurement took place was mixed thus arriving at a paradox.

(3.3) The other problem concerning separation we shall summarize is "de Broglie´s paradox". (Selleri and Tarozzi (1981) pp3-6). A box containing a quantum mechanical particle is divided into two parts in such a way that a non-zero part of the particle´s wave function lies in each half of the box. These two parts are now separated by moving the boxes far away from each other.

This situation may be described by assuming that the state vector of the system is a superposition $f = f_1 + f_2$ where f_1 and f_2 are essentially localised in disjoint distant regions. A measurement is now performed in one of these regions to determine if the particle is located there or not. Once the outcome of this measurement is known the probability of the particle being localised in the other region must be zero or one. But since the two regions are far apart a measurement performed in one can have no effect on the wave function in the other and it follows that the particle must have been localised in one of the regions before the measurement was performed. This is only possible if the initial state is a mixture of f_1 and f_2 which conflicts with the fact that $f_1 + f_2$ is pure and we have another paradox.

(3.4) To sum up we regard the crux of the matter as the conflict between the usual assumption that a linear combination $f + g$ of state vectors represents a pure state and the requirement from Einstein locality that such a combination should be a mixed state when the constituent parts of the system described by f and g are far apart.

Note that whether or not h represents a pure or mixed state is not simply a matter of interpretation. This is related to the set of all observables of the system. More precisely if for each vector h in the Hilbert space of the system we denote by P_h the projection onto the subspace spanned by h then for any two unit vectors f and g the density operators $P_{(f+g)/\sqrt{2}}$ and $\frac{1}{2}P_f + \frac{1}{2}P_g$ will give different expectation values for some hermitian operator A.

These paradoxes have motivated our study of sets of observables and our proposed solution will be found in chapter IV.

(3.5) General References.

Einstein Podolsky and Rosen (1935), de Broglie (1959), Jauch (1968) (pp185-7), Jammer (1974) (chapter 6), Selleri and Tarozzi (1981).

4. ALGEBRAIC QUANTUM MECHANICS.

In this section we give a brief review of the C*-algebra formulation of quantum mechanics. It will be assumed throughout this work that the reader is familiar with the basic theory of C* and von Neumann algebras (e.g. states and representations, the canonical cyclic representation associated with a state, the strong, strong* and weak topologies on $B(\mathcal{H})$, normal states on $B(\mathcal{H})$). All undefined terms appearing in this section may be found in Bratteli and Robinson (1979).

The C*-algebra approach to quantum theory developed from algebraic formulations due to Jordan, von Neumann and Wigner and to Segal. In the past two decades many applications to statistical mechanics and quantum field theory have been discovered. We shall begin with a brief summary of the physical interpretation and then discuss coherent and disjoint states.

(4.1) In the algebraic formulation of quantum mechanics it is usually assumed that the bounded observables of the system may be described by the Hermitian elements of some C*-algebra and that the states of the system are represented by states on the algebra. If $\mathcal{A}$ is the C*-algebra associated with such a system and if $A \in \mathcal{A}$ is Hermitian then for each state ω on $\mathcal{A}$, $\omega(A)$ is interpreted as the expectation value of the observable represented by A in the state represented by ω. The time evolution of the system is described by a homomorphism $\alpha : \mathbb{R} \longrightarrow \mathrm{aut}(\mathcal{A})$ from the group of real numbers to the group $\mathrm{aut}(\mathcal{A})$ of all *-automorphisms of $\mathcal{A}$. This homomorphism ω is usually assumed to be continuous in a suitable topology on $\mathcal{A}$. If ω is the initial state of the system then the state ω_t at time t is given by $\omega_t(A) = \omega(\alpha_t(A))$ for all $A \in \mathcal{A}$.

The algebraic approach may be used to describe a wide variety of physical systems e.g. quantum systems, classical systems and also quantum systems with superselection rules (see Primas and Muller-Herold (1978) section 3.9).

To obtain the basic Hilbert space formulation outlined in section 1, $\mathcal{A}$ is taken to be the C*-algebra $\mathbb{B}(\mathcal{H})$ of all bounded operators on the Hilbert space $\mathcal{H}$ describing the system and the states are identified with the normal states on $\mathbb{B}(\mathcal{H})$. The time evolution is described by defining α by $\alpha_t(A) = U_t^* A U_t$ for all $A \in \mathbb{B}(\mathcal{H})$ and all $t \in \mathbb{R}$ where U is the evolution group of the system.

Since this formulation also encompasses classical mechanics we do not expect to be able to form a superposition of any two states on an arbitrary C*-algebra .

(4.2) <u>Definition.</u> Two representations π_1 and π_2 of a C*-algebra are said to be disjoint if whenever τ_1 is a non-zero subrepresentation of π_1 and τ_2 is a non-zero subrepresentation of π_2 then τ_1 and τ_2 are not unitarily equivalent. Two states on a C*-algebra are called disjoint if their canonical cyclic representations are disjoint. Two pure states are said to be coherent if their (necessarily irreducible) canonical cyclic representations are unitarily equivalent .

A state ω on a C*-algebra $\mathcal{A}$ will be called a mixed state if it is not a pure state. If $\omega = a\omega_1 + b\omega_2$ where ω_1 and ω_2 are states on $\mathcal{A}$ and $a, b \in \mathbb{C}$ with $0 \leq a, b \leq 1$ and $a+b=1$ then we shall say that ω is a mixture of ω_1 and ω_2 .

Disjointness can also be characterised by the "cross-terms" in suitable representations of the algebra.

(4.3) <u>Lemma.</u> Let ω_1 and ω_2 be states on a C*-algebra $\mathcal{A}$, then ω_1 and ω_2 are disjoint if and only if for every representation $(\mathcal{H}, \pi)$ of $\mathcal{A}$ in which there are vectors $f, g \in \mathcal{H}$ with

$$\omega_1(A) = \langle f|\pi(A)f\rangle \qquad \text{and} \qquad \omega_2(A) = \langle g|\pi(A)g\rangle$$

for all $A \in \mathcal{A}$ then we have $\langle f|\pi(A)g\rangle = 0$.

(Proof: This is lemma 1 in Hepp(1972)).

(4.4) When two states ω_1 and ω_2 on a C*-algebra $\mathcal{A}$ are disjoint it follows from this lemma that in every representation π in which both ω_1 and ω_2 are vector states with corresponding vectors f and g , we have

$$\omega = |a|^2\omega_1 + |b|^2\omega_2$$

where ω is the state defined by

$$\omega(A) = \langle (af+bg)|\pi(A)(af+bg)\rangle \qquad (\forall A \in \mathcal{A})$$

where a and b are non-zero complex numbers with $|a|^2+|b|^2=1$. Hence physically the state ω obtained by forming the superposition $af+bg$ is equivalent to the mixture $|a|^2\omega_1+|b|^2\omega_2$ i.e. these states give the same expectation values for all observables.

(4.5) Suppose ω_1 and ω_2 are coherent pure states on a C*-algebra $\mathcal{A}$ and let $(\mathcal{H}_1, \pi_1)$ and $(\mathcal{H}_2, \pi_2)$ be the canonical cyclic representations associated with ω_1 and ω_2 respectively. Then there are vectors $f \in \mathcal{H}_1$ and $g \in \mathcal{H}_2$ with

$$\omega_1(A) = \langle f | \pi_1(A) f \rangle \quad \text{and} \quad \omega_2(A) = \langle g | \pi_2(A) g \rangle$$

for all $A \in \mathcal{A}$. Also there is a unitary operator $U : \mathcal{H}_2 \longrightarrow \mathcal{H}_1$ with $U^{-1} \pi_1(A) U = \pi_2(A)$ for all $A \in \mathcal{A}$. Now

$$\omega_2(A) = \langle Ug | \pi_1(A) Ug \rangle$$

for all $A \in \mathcal{A}$ so the states ω_1 and ω_2 can be described by vectors in the same Hilbert space $\mathcal{H}_1$. To form a state ω representing the superposition of ω_1 and ω_2 let a and b be non-zero complex numbers with $|a|^2 + |b|^2 = 1$. Define a state ω on $\mathcal{A}$ by

$$\omega(A) = \langle (af+bg) | \pi_1(A)(af+bg) \rangle, \qquad A \in \mathcal{A}.$$

Since $(\mathcal{H}_1, \pi_1)$ is irreducible it is easily verified that ω is pure.

Note that when $\omega_1 \neq \omega_2$ the state $|a|^2 \omega_1 + |b|^2 \omega_2$ represents a mixture (that it is not pure follows immediately from the algebraic definition of a pure state).

When $\mathcal{A}$ is the C*-algebra of all bounded operators on some Hilbert space then the usual notions of pure and mixed states coincide with those of the algebraic formulation provided we restrict our attention to normal states. In this case the density operator W in the Hilbert space $\mathcal{H}$ becomes identified with the state ω on $\mathbb{B}(\mathcal{H})$ defined by $\omega(A) = \mathrm{Tr}(WA)$ for all $A \in \mathcal{A}$.

(4.6) General References.

Bratteli and Robinson (1979), Emch (1972a), Gudder (1979), Guenin (1967), Roberts and Roepstorff(1969).

(4.7) Notes.

For a discussion of disjoint and coherent states see Hepp (1972) and Roberts and Roepstorff (1969). For a proof that the canonical cyclic representation associated with a pure state is irreducible see Bratteli and Robinson (1979) (p57). The result contained in (A5) may be used to show that the given definition of a superposition does indeed give a pure state.

5. GEOMETRIC SCATTERING THEORY.

In this section we present a review of some ideas from scattering theory which will be useful later. The geometric approach in which scattering states are related to the configuration space and evolution group of the system will be adopted. Throughout this section L^2 will denote the Hilbert space $L^2(\mathbb{R}^n)$ for some fixed n and E_Q and E_P will be the position and momentum spectral measures in $L^2(\mathbb{R}^n)$. All evolution groups will be defined in $L^2(\mathbb{R}^n)$.

(5.1) A scattering state of a physical system may be regarded as one which "propagates to infinity" as time increases i.e. the probability of a position measurement finding the system in some fixed bounded region of configuration space should be small for large time. Let V be the evolution group of a particle described by the Hilbert space L^2. Then $\|E_Q(R)V_t f\|^2$ is the probability of a position measurement at time t finding the particle in R where f is the unit vector representing the initial state. Hence such an f corresponds to a scattering state if $\|E_Q(R)V_t f\| \longrightarrow 0$ as $t \longrightarrow \infty$. A similar concept may be introduced for particles which "come from infinity".

(5.2) Definition. Let V be an evolution group in L^2 . An element f of L^2 will be called a positive scattering state of V if $\lim_{t\to\infty} \|E_Q(R)V_t f\| = 0$ for all bounded Borel sets R. Similarly an element f of L^2 will be called a negative scattering state of V if $\lim_{t\to-\infty} \|E_Q(R)V_t f\| = 0$ for all bounded Borel sets R.

(5.3) The set of all positive scattering states of an evolution group V is a closed subspace of L^2 which is invariant under each of the operators V_t $(t\in\mathbb{R})$. A similar property also holds for the set of all negative scattering states of V. (For a proof see Amrein (1981) p129.)

(5.4) Definition. In general we shall only consider evolution groups whose positive and negative scattering states coincide. In this case we talk of scattering states of V and denote the projection onto the subspace of all scattering states by $E_\infty(V)$.

(5.5) Example Let U be an evolution group such that $U_t \in L^\infty(P)$ for every $t\in\mathbb{R}$. Then each $f\in L^2$ is both a positive and a negative scattering state of U. So for such a U we have $E_\infty(U) = I$. In particular these results hold for a free particle evolution group . (For a proof see Amrein (1981) p132.)

The wave operators associated with a pair of evolution groups will now be defined. Physically V represents the time evolution of the system with interaction and U the free evolution.

(5.6) Definition. Let U and V be evolution groups then the wave operators $W_\pm$ are defined by

$$W_\pm = \underset{t \to \pm\infty}{\text{s-}\lim} V_t^* U_t E_\infty(U)$$

whenever these limits exist. These operators are said to be complete if the range of each is equal to the set of all scattering states of V.

This definition of completeness coincides with "asymptotic completeness in the geometric sense" as defined in Amrein (1981). Other definitions of completeness frequently require that the ranges of W_+ and W_- are equal to the continuous subspace of the Hamiltonian of the system. These definitions coincide for a large class of potentials in L^2. (See (5.9).) We now summarise the main properties of the wave operators.

(5.7) Theorem. Let V and U be evolution groups and assume that the wave operators

$$W_\pm = \underset{t \to \pm\infty}{\text{s-}\lim} V_t^* U_t E_\infty(U)$$

both exist and are complete, then

[1] Each of the operators W_+ and W_- is a partial isometry having initial domain equal to the set of all scattering states of U and final domain equal to the set of all scattering states of V;

[2] $$W_\pm^* = \underset{t \to \pm\infty}{\text{s-}\lim} U_t^* V_t E_\infty(V);$$

[3] $$W_\pm^* W_\pm = E_\infty(U), \quad W_\pm W_\pm^* = E_\infty(V);$$

[4] (The Intertwining Relations)

$$V_t W_\pm = W_\pm U_t, \quad U_t W_\pm^* = W_\pm^* V_t \qquad (t \in \mathbb{R}).$$

<u>Proof</u>. [1], [2] and [4] follow from propositions 4.1, 4.2 and 4.3 of Amrein, Jauch and Sinha (1977). Note that these authors take the set of scattering states of a self-adjoint operator H to be a subspace which is invariant under each of the operators exp(-itH) so by (5.3) their results are applicable to this situation. [3] is a standard property of partial isometries (e.g. Weidmann (1980) p85). ∎

(5.8) The physical interpretation of the existence of the wave operators is that the actual evolution (with respect to the evolution group V) of a scattering state can be approximated by the free evolution (with respect to U) of some other state. More precisely each scattering state f of V belongs to the final domain of W_+ and hence to the initial domain of W_+^* so $W_+^* f$ is a scattering state of U. This implies $E_\infty(U)W_+^* f = W_+^* f$ so

$$\begin{aligned}
\lim_{t\to\infty} \| V_t f - U_t(W_+^* f)\| &= \lim_{t\to\infty} \| V_t f - U_t E_\infty(U) W_+^* f \| \\
&= \lim_{t\to\infty} \| f - V_t^* U_t E_\infty(U) W_+^* f \| \\
&= \| f - W_+ W_+^* f \| \\
&= \| f - E_\infty(V) f \| \\
&= 0 .
\end{aligned}$$

Hence for large t the evolution of the scattering state f of V can be approximated by the free evolution of the scattering state $W_+^* f$ of U. The scattering state $W_-^* f$ of U gives a similar approximation for large negative time,

$$\lim_{t\to-\infty} \| V_t f - U_t(W_-^* f)\| = 0 .$$

The following result shows that the wave operators exist and are complete for many Hamiltonians in L^2.

(5.9) Let v be a function from $\mathbb{R}^n$ to $\mathbb{R}$ such that for all $x \in \mathbb{R}^n$,

$$v(x) = (1+|x|)^{-q}(V_1(x) + V_2(x))$$

where $V_1 \in L^\infty(\mathbb{R}^n)$, $V_2 \in L^p(\mathbb{R}^n)$ and p and q are integers with $p \geqslant 2$, $p \geqslant n/2$ and $q \geqslant 0$. Let v(Q) be the multiplication operator on L^2 defined by the function v, then the operator $H = H_0 + v(Q)$ is self-adjoint where H_0 is a free particle Hamiltonian (2.5). Let

$$V_t = \exp(tH/i\hbar), \quad U_t = \exp(tH_0/i\hbar)$$

then the wave operators $W_\pm = \underset{t\to\pm\infty}{\text{s-lim}}\, V_t^* U_t$ exist and are complete. Moreover the set of all scattering states of V coincides with the continuous subspace of H (this is defined to be the orthogonal complement of the subspace generated by the eigenvectors of H).
(For a proof see Amrein (1981) p176.)

(5.10) General References.

Textbooks: Amrein (1981), Amrein, Jauch and Sinha (1977)

Putnam (1967) chapter V, Weidmann (1980) chapter 11.

Review Article: Enss (1981).

CHAPTER II

ASYMPTOTIC PROPERTIES OF STATE VECTORS.

6. ASYMPTOTIC LOCALISATION OF STATES.

It is well known that a wave function of a free quantum mechanical system spreads with time and that the support of such a function can change instantaneously from a bounded to an unbounded set. For a precise discussion of these properties we refer the reader to Hegerfeldt and Ruijsennars (1980). In this section it will be shown that despite these difficulties it is possible to introduce an approximate localisation of certain wave functions. This notion will then be used to define spatially separating states and spatially separating systems.

In classical mechanics if a free particle with configuration space $\mathbb{R}^n$ is initially situated at the origin and has momentum in some set S then at time t the position of the particle will lie in the set $tS = \{tx: x \in S\}$. If the initial velocity is not known precisely then in general the "length" of the set tS in which the particle is predicted to lie will spread. More generally the probability that the particle lies in the set tS at time t is equal to the probability that its velocity belongs to the set S. We now show that the probability of a measurement finding a quantum particle in the set tS at time t, converges as $t \to \infty$ to the probability that the wave function has "velocity" in the set S.

Throughout this section L^2 will denote the Hilbert space $L^2(\mathbb{R}^n)$ for some fixed n, and E_Q and E_P will denote the position and momentum spectral measures in this space.The velocity of a quantum mechanical particle of mass m may be defined in terms of the spectral measure $S \mapsto E_P(mS)$ on the Borel sets of $\mathbb{R}^n$. We shall interpret $\|E_P(mS)f\|^2$ as the probability that a measurement will find the velocity of the particle in S when f is the unit vector describing the state. If this probability is equal to unity then we shall say the velocity of the particle lies in S.

(6.1) <u>Theorem.</u> Let U be the free particle evolution group of a particle of mass m. Then for all Borel sets S and all elements f of L^2,

$$\lim_{t\to\pm\infty} \| E_Q(tS)\, U_t f \| = \| E_P(mS) f \| .$$

<u>Proof.</u> For $t \neq 0$ let $D_{m/t}$ be the operator defined by

$$(D_{m/t} f)(x) = \left|\frac{m}{t}\right|^{n/2} f\left(\frac{mx}{t}\right)$$

for all $f \in L^2$ and all $x \in \mathbb{R}^n$. Also let

$$C_t^\circ = a(t)\, D_{m/t} \exp\left(\frac{it}{2m\hbar} Q^2\right) F$$

where F is the Fourier transform in L^2 and

$$a(t) = \exp\left(-\frac{in\pi t}{4|t|}\right) .$$

Then by a result of Dollard (A3.4)

$$U_t = C_t^\circ \exp\left(\frac{im}{2t\hbar} Q^2\right)$$

and $$\lim_{t\to\pm\infty} \| U_t f - C_t^\circ f \| = 0 \qquad \text{for all } f \in L^2.$$

Now let S be a Borel set and let $f \in L^2$ then $\exp\left(\frac{it}{2m\hbar} Q^2\right)$ is unitary and commutes with every $E_Q(S)$ so from the definition of C_t°,

$$\| E_Q(S) D_{m/t}^{-1} C_t^\circ f \| = \| E_Q(S)\, a(t) \exp\left(\frac{it}{2m\hbar} Q^2\right) F f \|$$

$$= \quad \| E_Q(S) F f \|$$

$$= \quad \| F^{-1} E_Q(S) F f \|$$

$$= \quad \| E_P(S) f \| .$$

Now $D_{m/t}^{-1} = D_{t/m}$ and

$$\begin{aligned} & \Big| \| E_Q(S) D_{t/m} U_t f \| - \| E_P(S) f \| \Big| \\ = \quad & \Big| \| E_Q(S) D_{t/m} U_t f \| - \| E_Q(S) D_{t/m} C_t^{\circ} f \| \Big| \\ \leqslant \quad & \| E_Q(S) D_{t/m} (U_t - C_t^{\circ}) f \| \\ \leqslant \quad & \| U_t f - C_t^{\circ} f \| \\ \to \quad & 0 \qquad\qquad (t \to \pm\infty) \end{aligned}$$

by the result from (A3.4) quoted above. Hence

$$\lim_{t \to \pm\infty} \| E_Q(S) D_{t/m} U_t f \| = \| E_P(S) f \| .$$

But

$$\begin{aligned} \| E_Q(S) D_{t/m} U_t f \|^2 &= \int \chi_S(x) \, |\tfrac{t}{m}|^n \, |(U_t f)(tx/m)|^2 \, dx \\ &= \int \chi_S(tx/m) \, |(U_t f)(x)|^2 \, dx \\ &= \| E_Q(\tfrac{t}{m} S) U_t f \|^2 \end{aligned}$$

and since S and f are arbitrary we may replace S by mS and deduce

$$\lim_{t \to \pm\infty} \| E_Q(tS) U_t f \| = \| E_p(mS) f \| . \qquad \blacksquare$$

(6.2) We now discuss the physical interpretation of this result for a free particle whose initial state is described by a unit vector f in L^2. Here $\| E_Q(tS) U_t f \|^2$ is the probability that a position measurement at time t will find the particle in the subset tS of configuration space $\mathbb{R}^n$. We shall call $\lim_{t \to +\infty} \| E_Q(tS) U_t f \|^2$ the probability that the particle is localised in tS for large time. Also $\| E_P(mS) f \|^2$ is the probability that a velocity measurement at t = 0 (or indeed at any time since $\| E_P(mS) f \|^2 = \| E_P(mS) U_t f \|^2$) will give a value in S. Hence the system is localised in tS for large time if and only if its velocity lies in the set S.

In Wan and McLean (1983a,b) an element f of L^2 is said to be asymptotically localizable in tS if S is an interval in $\mathbb{R}^n$ with $\lim_{t\to+\infty} \| E_Q(tS)\, U_t f \|^2 = 1$.

A similar result is obtained if the particle is approximately free at large time. By assuming the existence of suitable wave operators the actual state of the system can be approximated by a freely evolving state and the following theorem is obtained.

(6.3) Theorem. Let V be an evolution group in L^2 and assume that the wave operators

$$W_\pm = \underset{t\to\pm\infty}{s\text{-}\lim}\ V_t^* U_t$$

both exist and are complete. Then for all Borel sets S and all scattering states f of V,

$$\lim_{t\to\pm\infty} \| E_Q(tS) V_t f \| = \lim_{t\to\pm\infty} \| E_P(mS) V_t f \|$$

$$= \| E_P(mS) W_\pm^* f \| = \| W_\pm E_P(mS) W_\pm^* f \| .$$

Proof. Recall that the adjoint of W_+ is given by

$$W_+^* = \underset{t\to+\infty}{s\text{-}\lim}\ U_t^* V_t E_\infty(V)$$

and that $W_+ W_+^* = E_\infty(V)$, $W_+^* W_+ = I$.

Let f be a scattering state of V then $f = E_\infty(V)f$ and since U_t^* is unitary and commutes with $E_P(mS)$ we have

$$\| E_P(mS) V_t f \| = \| E_P(mS) V_t E_\infty(V) f \|$$

$$= \| E_P(mS) U_t^* V_t E_\infty(V) f \|$$

$$\to \| E_P(mS) W_+^* f \| \qquad (t\to+\infty) .$$

By the last theorem $\| E_P(mS) (W_+^* f) \| = \lim_{t\to+\infty} \| E_Q(tS) U_t (W_+^* f) \|$

and to simplify this limit observe that

$$\begin{aligned} \| E_Q(tS) V_t f - E_Q(tS) U_t W_+^* f \| &\leq \| V_t f - U_t W_+^* f \| \\ &= \| f - V_t^* U_t W_+^* f \| \\ &\to \| f - W_+ W_+^* f \| \qquad (t \to +\infty) \\ &= \| f - E_\infty(V) f \| \\ &= 0 . \end{aligned}$$

This now establishes that

$$\lim_{t \to +\infty} \| E_Q(tS) V_t f \| = \| E_P(mS) W_+^* f \| = \lim_{t \to +\infty} \| E_P(mS) V_t f \| .$$

Finally since W_+ is a partial isometry with initial domain L^2 we have $\| E_P(mS) W_+^* f \| = \| W_+ E_P(mS) W_+^* f \|$.

This proves the result for $t \to +\infty$. The proof of the other case is identical except for an obvious change of sign. ∎

(6.4) If $\lim_{t \to +\infty} \| E_P(mS) V_t f \|^2 = 1$ then the probability of a velocity measurement giving a value in S is close to 1 for large time. In this case we may say that the velocity of the particle after scattering lies in S. Note also that the actual state $V_t f$ can be approximated for large t by a state $U_t W_+^* f$ in which the velocity actually lies in the set S. Hence we shall say that the system is localised in tS for large time if and only if the velocity after scattering lies in S.

7. SPATIAL SEPARATION.

The results of the last section show that the existence of a suitable wave operator enables us to regard a scattering state as approximately localised in some region tS for large values of time t where the set S is determined by the velocity (or momentum) of the particle. We shall now use this asymptotic localisation to define spatially separating states and spatially separating systems.

Intuitively two states are spatially separating if they are localised in disjoint regions for all sufficiently large time. In quantum mechanics we shall say that two normalised wave functions are spatially separating if the probability that they can be localised in disjoint regions is close to 1 for large time.

(7.1) <u>Definition</u>. Let V be an evolution group in $L^2(\mathbb{R}^n)$. We shall say that two scattering states f and g of V are spatially separating with respect to V if there are disjoint Borel sets R and S in $\mathbb{R}^n$ with

$$\lim_{t\to+\infty} \| E_Q(tR) V_t f \| = \| f \| ,$$

and

$$\lim_{t\to+\infty} \| E_Q(tS) V_t g \| = \| g \| .$$

(The name asymptotically separable states was used in papers by Wan and McLean (1983a,b,d) to denote what we have called spatially separating states). We have some immediate corollaries to the results of the last section.

(7.2) Corollary. Let U be the free particle evolution group in $L^2(\mathbb{R}^n)$ for a particle of mass m. Let V be another evolution group in $L^2(\mathbb{R}^n)$ such that the wave operator

$$W_+ = \underset{t \to +\infty}{s\text{-}\lim} V_t^* U_t$$

exists and is complete. If f and g are scattering states of V then these are equivalent

[1] f and g are spatially separating with respect to V;

[2] there are disjoint Borel sets R and S with

$$\lim_{t \to +\infty} \| E_P(mR) V_t f \| = \| f \| \quad \text{and} \quad \lim_{t \to +\infty} \| E_P(mS) V_t g \| = \| g \| ;$$

[3] there are disjoint Borel sets R and S with

$$E_P(mR) W_+^* f = W_+^* f \quad \text{and} \quad E_P(mS) W_+^* g = W_+^* g ;$$

[4] $W_+^* f$ and $W_+^* g$ are spatially separating with respect to U .

Proof. This follows from the definition (7.1) of spatial separation and (6.3). ∎

(7.3) Corollary. Two elements f and g of $L^2(\mathbb{R}^n)$ are spatially separating with respect to the free particle evolution group of a particle of mass m if and only if there are disjoint Borel sets R and S with

$$E_P(mR) f = f \quad \text{and} \quad E_P(mS) g = g .$$

<u>Proof</u>. Take $V = U$ in the last corollary then $W_{+} = I = W_{+}^{*}$. ∎

Hence we may say that the two states f and g are spatially separating (with respect to a free particle evolution group) if and only if they correspond to disjoint momentum values (or to disjoint velocity values).

We may regard two particles each with configuration space $\mathbb{R}^n$ as spatially separating if the individual particles can be approximately localised in disjoint regions for large time. Before giving a formal definition we show that the results of the last section may be extended to describe the localisation of the individual particles making up a two particle system.

First we introduce the <u>notation which will be used for the remainder of this section</u>.

We shall investigate a two particle system where the first particle has mass m_1 and the second has mass m_2. We may assume $m_1 \neq m_2$ and that the particles are therefore distinguishable. For $i = 1,2$, U^i will denote the free particle evolution group in for a particle of mass m_i. Then the free evolution group U in $L^2(\mathbb{R}^{2n})$ describing the combined system is given by

$$U_t = U_t^1 \otimes U_t^2 \qquad (\forall t \in \mathbb{R}).$$

E_Q and E_P will denote the position and momentum spectral measures in $L^2(\mathbb{R}^n)$ and $E_{Q\otimes Q}$ and $E_{P\otimes P}$ the position and momentum spectral measures in $L^2(\mathbb{R}^{2n})$. We introduce the spectral measures $E_{Q\otimes I}$, $E_{I\otimes Q}$,

$E_{P\otimes I}$, and $E_{I\otimes P}$ as defined in (2.6) to describe the positions and momenta of the individual systems.

(7.4) Theorem. For all $f \in L^2(\mathbb{R}^{2n})$ and all Borel sets S in $\mathbb{R}^n$

$$\lim_{t\to\pm\infty} \| E_{Q\otimes I}(tS)U_t f \| = \| E_{P\otimes I}(m,S) f \|$$

and

$$\lim_{t\to\pm\infty} \| E_{I\otimes Q}(tS)U_t f \| = \| E_{I\otimes P}(m,S) f \| .$$

Proof. We prove the result for $Q\otimes I$ and $t\to+\infty$ only since the other cases follow from an identical argument.

Let $f \in L^2(\mathbb{R}^{2n})$ and let $\{e_r : r \in \mathbb{N}\}$ be an orthonormal basis for $L^2(\mathbb{R}^n)$. Then for some sequence g_r in $L^2(\mathbb{R}^n)$ we have

$$f = \sum_{r=1}^{\infty} g_r \otimes e_r .$$

Note that the set

$$\{ (E_P(m,S)g_r) \otimes e_r \ : \ r \in \mathbb{N} \}$$

is orthogonal and that for every $t \in \mathbb{R}$ the following set is also orthogonal:

$$\{ (E_Q(tS)U_t^1 g_r) \otimes (U_t^2 e_r) \ : \ r \in \mathbb{N} \} .$$

Now using (6.1) we have

$$\begin{aligned}
\| E_{P\otimes I}(m,S) f \|^2 &= \| \sum_{r=1}^{\infty} (E_P(m,S)\otimes I)(g_r \otimes e_r) \|^2 \\
&= \| \sum_{r=1}^{\infty} (E_P(m,S)g_r)\otimes e_r \|^2 \\
&= \sum_{r=1}^{\infty} \| E_P(m,S)g_r \|^2 \\
&= \sum_{r=1}^{\infty} \lim_{t\to+\infty} \| E_Q(tS)U_t^1 g_r \|^2 \\
&= \lim_{t\to+\infty} \sum_{r=1}^{\infty} \| E_Q(tS)U_t^1 g_r \|^2 \\
&= \lim_{t\to+\infty} \sum_{r=1}^{\infty} \| (E_Q(tS)U_t^1 g_r) \otimes U_t^2 e_r \|^2 \\
&= \lim_{t\to+\infty} \| \sum_{r=1}^{\infty} (E_Q(tS)U_t^1 g_r) \otimes U_t^2 e_r \|^2
\end{aligned}$$

$$= \lim_{t \to +\infty} \left\| \sum_{r=1}^{\infty} \left(E_Q(tS) \otimes I \right) U_t \left(g_r \otimes e_r \right) \right\|^2$$

$$= \lim_{t \to +\infty} \| E_{Q\otimes I}(tS) \, U_t f \, \|^2.$$

To justify interchanging the order of the limit and summation in the above argument note that for each $t \in \mathbb{R}$

$$\| E_Q(tS) \, U_t^1 g_r \|^2 \leq \| g_r \|^2 = \| g_r \otimes e_r \|^2,$$

and also that $\sum_{r=1}^{\infty} \| g_r \otimes e_r \|^2 = \| f \|^2$ so by the Weierstrass M-test (Apostol (1974) p223) $\sum_r \| E_Q(tS) \, U_t^1 g_r \|^2$ converges uniformly (on $\mathbb{R}$) and the result of (A1) may be applied. ∎

(7.5) Theorem. Let V be an evolution group in $L^2(\mathbb{R}^{2n})$ and assume that the wave operators

$$W_\pm = \underset{t \to \pm\infty}{s\text{-}\lim} \; V_t^* U_t$$

both exist and are complete. Then for all Borel sets S in $\mathbb{R}^n$ and all scattering states f of V

$$\lim_{t \to \pm\infty} \| E_{Q\otimes I}(tS) V_t f \| = \lim_{t \to \pm\infty} \| E_{P\otimes I}(m_1 S) V_t f \|$$

$$= \| E_{P\otimes I}(m_1 S) \, W_\pm^* f \|$$

$$= \| W_\pm E_{P\otimes I}(m_1 S) \, W_\pm^* f \| .$$

These equalities remain valid when $Q\otimes I$, $P\otimes I$ and m_1 are replaced by $I\otimes Q$, $I\otimes P$ and m_2 respectively.

Proof. Since $U_t^* = (U_t^1)^* \otimes (U_t^2)^*$ is unitary and commutes with $E_{P\otimes I}(m_1 S) = E_P(m_1 S) \otimes I$ and also with $E_{I\otimes P}(m_2 S) = I \otimes E_P(m_2 S)$, the proof of this result follows from (7.4) by using an identical argument to that employed in the proof of (6.3). ∎

(7.6) Definition. Let V be an evolution group in $L^2(\mathbb{R}^{2n})$ and let f be a scattering state of V. Then we shall say that the individual systems are spatially separating (with respect to V) in the state f if there are disjoint Borel sets R and S in $\mathbb{R}^n$ with

$$\lim_{t\to+\infty} \| E_{Q\otimes I}(tR) V_t f \| = \| f \| = \lim_{t\to+\infty} \| E_{I\otimes Q}(tS) V_t f \| .$$

We have some direct corollaries to the last two theorems:

(7.7) Corollary. Let $f \in L^2(\mathbb{R}^{2n})$ then the systems are spatially separating (with respect to the free evolution group U) in the state f if and only if there are disjoint Borel sets R and S in $\mathbb{R}^n$ with

$$E_{P\otimes I}(m_1 R) f = f = E_{I\otimes P}(m_2 S) f .$$

Proof. This follows immediately from (7.4) and (7.6). ∎

Hence we may say that two particles are spatially separating if and only if they have disjoint velocity values in the state f.

(7.8) Corollary. Let f be a scattering state of an evolution group V in $L^2(\mathbb{R}^{2n})$ and assume that the wave operator

$$W_+ = \underset{t\to+\infty}{s\text{-}\lim}\ V_t^* U_t$$

exists and is complete. Then these are equivalent,

[1] the systems are spatially separating (with respect to V) in the state f;

[2] there are disjoint Borel sets R and S with

$$\lim_{t\to+\infty} \| E_{P\otimes I}(m_1 R) V_t f \| = \| f \| = \lim_{t\to+\infty} \| E_{I\otimes P}(m_2 S) V_t f \| ;$$

[3] there are disjoint Borel sets R and S with

$$E_{P\otimes I}(m_1 R)\, W_+ f \;=\; f \;=\; E_{I\otimes P}(m_2 S)\, W_+ f\,;$$

[4] the systems are spatially separating (with respect to U) in the state $W_+ f$.

Proof. This follows immediately from (7.5) and (7.6). ∎

CHAPTER III. ASYMPTOTIC PROPERTIES OF OBSERVABLES.

8. A C*-ALGEBRA GENERATED BY LOCAL OBSERVABLES.

In this section we study a "quasi-local" algebra of operators associated with a quantum system. This algebra has a similar mathematical structure to the C*-algebras of quasi-local observables introduced in quantum field theory by Haag and Kastler (1964) and arises in a similar manner by considering the nature of local measurements. In recent years this algebraic approach has been applied to tackle quantum mechanical measurement problems (Hepp (1972), Emch (1972b), Whitten-Wolfe and Emch (1976)). The essential difference in our approach is that the algebras are constructed to describe a single non-relativistic quantum particle.

Throughout this section L^2 will denote the Hilbert space $L^2(\mathbb{R}^n)$ for some fixed n and E_Q and E_P will be the usual position and momentum spectral measures in this space.

Hermitian operators of the form $E_Q(S)AE_Q(S)$ where $A \in \mathbb{B}(L^2)$ and S is a bounded Borel set will be called local observables. Such operators will be interpreted as bounded observables which can be measured by an apparatus of "size S". For a discussion of the physical motivation behind this idea the reader may consult Wan and Jackson (1983) and Wan, McKenna and Jackson (1983). We shall discuss some mathematical properties of the C*-algebra generated by these observables.

(8.1) Definition. For each Borel set S in $\mathbb{R}^n$ define $\mathscr{A}(S)$ by

$$\mathscr{A}(S) = \{A \in \mathbb{B}(L^2) : A = E_Q(S) X E_Q(S) \text{ for some } X \in \mathbb{B}(L^2)\}.$$

We also define $\mathscr{A}_L$ to be the set of all operators A in $\mathbb{B}(L^2)$ with $A = E_Q(S)XE_Q(S)$ for some bounded Borel set S and some $X \in \mathbb{B}(L^2)$. The closure of $\mathscr{A}_L$ in the operator norm will be denoted by $\bar{\mathscr{A}}_L$.

(8.2) Lemma. Let S be a Borel set then $\mathscr{A}(S)$ is a C*-subalgebra of $\mathbb{B}(L^2)$ and $A \in \mathscr{A}(S)$ if and only if $A = E_Q(S)AE_Q(S)$.

Proof. It is easily verified that $\mathscr{A}(S)$ is a *-subalgebra of $\mathbb{B}(L^2)$. If $A = E_Q(S) A E_Q(S)$ then clearly $A \in \mathscr{A}(S)$ and conversely if $A = E_Q(S) X E_Q(S)$ for some $X \in \mathbb{B}(L^2)$ then

$$\begin{aligned} E_Q(S)\, A\, E_Q(S) &= E_Q(S)\,(E_Q(S) X E_Q(S))\, E_Q(S) \\ &= E_Q(S)\, X\, E_Q(S) \\ &= A \end{aligned}$$

Finally $\mathscr{A}(S)$ is closed since if A_r is a sequence in $\mathscr{A}(S)$ converging to $A \in \mathbb{B}(L^2)$ then $A_r = E_Q(S) A_r E_Q(S)$ for every r and

$$\| A - E_Q(S) A E_Q(S) \| = \lim_{r\to\infty} \| A_r - E_Q(S) A_r E_Q(S) \| = 0$$

so $A \in \mathscr{A}(S)$. ∎

(8.3) Lemma. Let $A \in \mathbb{B}(L^2)$ and let S_r be an increasing sequence of Borel sets converging to $\mathbb{R}^n$. Then $E_Q(S_r)AE_Q(S_r)$ converges strongly to A.

Proof. Since E_Q is a spectral measure the sequence $E_Q(S_r)$ converges strongly to the identity operator I and the result now follows from a multiplicative property of strong convergence (Weidmann (1980) p80). ∎

In particular (8.3) implies that the expectation value of an observable A in any normal state can be approximated by the expectation value of some local observable $E_Q(S)AE_Q(S)$ where S is bounded.

(8.4) Corollary. Let ω be a normal state on $\mathbb{B}(L^2)$ and let $A \in \mathbb{B}(L^2)$. Then for every $\varepsilon > 0$ there is a bounded Borel set S in $\mathbb{R}^n$ with

$$|\omega(A) - \omega(E_Q(S)AE_Q(S))| < \varepsilon .$$

Proof. Let S_r be an open ball of radius r centred at the origin in $\mathbb{R}^n$ then by the lemma $E_Q(S_r)AE_Q(S_r)$ converges strongly to A. Hence (e.g. Bratteli and Robinson (1979) p67) $E_Q(S_r)AE_Q(S_r)$ converges to A in the σ-strong topology and therefore in the σ-weak topology. The result now follows from the fact that a normal state is σ-weakly continuous (Bratteli and Robinson (1979) p67). ∎

(8.5) Theorem. $\mathcal{A}_L$ is a proper $*$-subalgebra of $\mathbb{B}(L^2)$. Also $\mathcal{A}_L$ is irreducible and the von Neumann algebra it generates is $\mathbb{B}(L^2)$.

Proof. Let $A, B \in \mathscr{A}_L$ then there are bounded Borel sets R and S with $A = E_Q(R) A E_Q(R)$ and $B = E_Q(S) B E_Q(S)$. Now $R \cup S$ is bounded and

$$A + B = E_Q(R \cup S)(A+B) E_Q(R \cup S),$$

$$AB = E_Q(R \cup S)(A E_Q(R) E_Q(S) B) E_Q(R \cup S).$$

It is now clear that $\mathscr{A}_L$ is a *-subalgebra of $\mathbb{B}(L^2)$. If $\mathscr{A}_L = \mathbb{B}(L^2)$ then $I = E_Q(R) I E_Q(R) = E_Q(R)$ for some bounded Borel set R which gives a contradiction, so $\mathscr{A}_L$ is a proper *-subalgebra.

Since a von Neumann algebra is closed in the strong topology it follows from (8.3) that $\mathbb{B}(L^2)$ is the von Neumann algebra generated by $\mathscr{A}_L$. Now $\mathscr{A}_L'' = \mathbb{B}(L^2)$ so

$$\mathscr{A}_L' = \mathscr{A}_L''' = \mathbb{B}(L^2)' = \mathbb{C} I$$

and $\mathscr{A}_L$ is therefore irreducible. ∎

It is easily verified that the C*-algebra $\bar{\mathscr{A}}_L$ has a quasi-local structure in the following sense,

(8.6) Theorem. For every bounded Borel set S, $\mathscr{A}(S)$ is a C*-algebra with identity and the union of these algebras is norm dense in $\bar{\mathscr{A}}_L$. If R and S are bounded Borel sets then

[1] if $R \subseteq S$ then $\mathscr{A}(R) \subseteq \mathscr{A}(S)$,

[2] if R and S are disjoint then $[A,B] = 0$ for all $A \in \mathscr{A}(R)$ and all $B \in \mathscr{A}(S)$.

<u>Proof.</u> $\mathcal{A}(S)$ is a C*-algebra by (8.2) and $E_Q(S)$ is an identity for $\mathcal{A}(S)$. The union is norm dense in $\bar{\mathcal{A}}_L$ by definition of $\bar{\mathcal{A}}_L$ (8.1).

If $R \subseteq S$ then $E_Q(R) A E_Q(R) = E_Q(S) E_Q(R) A E_Q(R) E_Q(S)$ for all $A \in \mathbb{B}(L^2)$ so $\mathcal{A}(R) \subseteq \mathcal{A}(S)$.

If R and S are disjoint then $E(R)E(S) = O$ so

$$E(R) A E(R) E(S) B E(S) = O = E(S) B E(S) E(R) A E(R)$$

for all $A, B \in \mathbb{B}(L^2)$. Hence $[A,B] = 0$ for all $A \in \mathcal{A}(R)$ and all $B \in \mathcal{A}(S)$. ∎

For a definition of a quasi-local algebra see Brattelі and Robinson (1979) or Hepp (1972). Our next theorem gives some useful conditions for deciding if a given operator belongs to $\bar{\mathcal{A}}_L$.

(8.7) <u>Theorem.</u> Let $A \in \mathbb{B}(L^2)$ then these are equivalent

[1] $A \in \bar{\mathcal{A}}_L$;

[2] there is a sequence R_r of bounded Borel sets in $\mathbb{R}^n$ such that $E_Q(R_r)AE_Q(R_r)$ converges to A in the operator norm;

[3] for every increasing sequence S_r of open balls converging to $\mathbb{R}^n$ the sequence $E_Q(S_r)AE_Q(S_r)$ converges to A in the operator norm.

<u>Proof.</u> We let E denote the spectral measure E_Q throughout.

([1] $\Rightarrow$ [2]) Suppose $A \in \bar{\mathcal{A}}_L$ then there is a sequence A_r in $\mathcal{A}_L$ converging in the operator norm to A and for each $r \in \mathbb{N}$,

$A_r = E(R_r) A_r E(R_r)$ for some bounded Borel set R_r. Now

$$\begin{aligned} & \| E(R_r) A E(R_r) - A \| \\ \leqslant\ & \| E(R_r) A E(R_r) - E(R_r) A_r E(R_r) \| + \| A_r - A \| \\ \leqslant\ & 2 \| A_r - A \| \\ \to\ & 0 \qquad\qquad (r \to \infty) \end{aligned}$$

so [2] holds.

([2] $\Rightarrow$ [3]) Assume [2] holds and define $B_k = \bigcup_{r=1}^{k} R_r$ then B_k is an increasing sequence of bounded Borel sets and $R_k \subseteq B_k$ for each $k \in \mathbb{N}$. Let S_r be an increasing sequence of open balls converging to $\mathbb{R}^n$ then we can extract a strictly increasing subsequence S_{r_k} $(k \in \mathbb{N})$ such that

$$B_k \subseteq S_{r_k} \qquad\qquad \text{for every } k \in \mathbb{N}.$$

Now $R_k \subseteq S_{r_k}$ so $E(S_{r_k}) E(R_k) = E(R_k)$ giving

$$\begin{aligned} & \| E(S_{r_k}) A E(S_{r_k}) - A \| \\ \leqslant\ & \| E(S_{r_k}) A E(S_{r_k}) - E(S_{r_k}) E(R_k) A E(R_k) E(S_{r_k}) \| \\ & \qquad + \| E(R_k) A E(R_k) - A \| \\ \leqslant\ & 2 \| E(R_k) A E(R_k) - A \| . \end{aligned}$$

Hence a subsequence of $E(S_r) A E(S_r)$ converges to A. To show that the sequence itself converges to A observe that for $r_k \leq r$ similar inequalities to the above give

$$\begin{aligned} \| E(S_r) A E(S_r) - A \| & \leq 2 \| E(S_{r_k}) A E(S_{r_k}) - A \| \\ & \leq 4 \| E(R_k) A E(R_k) - A \| \end{aligned}$$

and this implies $E(S_r) A E(S_r)$ converges to A.

Finally [3] $\Rightarrow$ [1] is obvious from the definition of $\bar{\mathscr{A}}_L$. ∎

It is easily shown that property [3] in (8.7) also holds when the sequence of open balls is replaced by any sequence S_r having the property that there is an increasing sequence R_r of open balls converging to $\mathbb{R}^n$ with $R_r \leq S_r$ for each $r \in \mathbb{N}$. So in particular we may take S_r to be a sequence of closed balls or a sequence of bounded intervals in $\mathbb{R}^n$. The following example shows that [3] need not hold for an arbitrary sequence converging to $\mathbb{R}^n$.

(8.8) Example. Define R_r and S by

$$R_r = \{x \in \mathbb{R}^n : 1/r \leq |x| \leq r\} \cup \{0\} \qquad (r \in \mathbb{N})$$

$$S = \{x \in \mathbb{R}^n : |x| \leq 1\}$$

Then $E_Q(S) \in \widetilde{\mathcal{A}}_L$ and R_r is an increasing sequence converging to $\mathbb{R}^n$ but $E_Q(R_r)E_Q(S)E_Q(R_r)$ does not converge in the operator norm.

To prove this note that for each $r \in \mathbb{N}$

$$\begin{aligned} E_Q(R_r)E_Q(S)E_Q(R_r) &= E_Q(S \cap R_r) \\ &= E_Q(\{x \in \mathbb{R}^n : 1/r \leq |x| \leq 1\}) \end{aligned}$$

Now this is a decreasing sequence of projections whose terms are all distinct and hence does not converge in the operator norm. (If M and N are projections with $M < N$ then $N - M$ is a non-zero projection so $\|N - M\| = 1$. This implies that a monotone sequence of projections converges in the operator norm if and only if it is eventually constant.)

(8.9) Theorem. $\widetilde{\mathcal{A}}_L$ is a proper C*-subalgebra of $\mathcal{B}(L^2)$ which does not contain the identity operator. $\widetilde{\mathcal{A}}_L$ contains all compact operators and $\widetilde{\mathcal{A}}_L \neq \mathcal{A}_L$.

Proof. Let E denote the spectral measure E_Q .

Let S_r denote an open ball of radius r centred at the origin in $\mathbb{R}^n$. If $I \in \bar{\mathcal{A}}_L$ then by (8.7) $E(S_r) \to I$ in the operator norm which gives a contradiction since the sequence is not eventually constant (c.f. the remark at the end of (8.8)). Hence $I \notin \bar{\mathcal{A}}_L$.

Let f be a unit vector in L^2 and let P_f be the projection onto the subspace spanned by f . We begin by showing that $P_f \in \bar{\mathcal{A}}_L$. Define $P_r \in \mathbb{B}(L^2)$ by

$$P_r g = \langle E(S_r) f \mid g \rangle E(S_r) f \qquad (r \in \mathbb{N})$$

then $P_r = E(S_r)\, P_r\, E(S_r)$ and

$$\begin{aligned} \| P_r - P_f \| &= \sup_{\|g\|=1} \| P_m g - P_f g \| \\ &= \sup_{\|g\|=1} \| \langle E(S_r) f \mid g \rangle E(S_r) f - \langle f \mid g \rangle f \| \\ &\leq \sup_{\|g\|=1} \left(\| \langle E(S_r) f - f \mid g \rangle E(S_r) f \| + \| \langle f \mid g \rangle (E(S_r) f - f) \| \right) \\ &\leq \| E(S_r) f - f \| \, \| E(S_r) f \| + \| E(S_r) f - f \| \\ &\to 0 \qquad (m \to \infty) \end{aligned}$$

since $E(S_r) \to I$ strongly. It follows that $\bar{\mathcal{A}}_L$ contains all one-dimensional projections and hence all finite-dimensional projections. Now by the spectral theorem for compact operators (Weidmann (1980) p166) $\bar{\mathcal{A}}_L$ contains all compact Hermitian operators. To complete the proof that $\bar{\mathcal{A}}_L$ contains all compact operators observe that an arbitrary compact operator is of the form A + iB where A and B are compact and Hermitian.

Finally we show that $\bar{\mathcal{A}}_L \neq \mathcal{A}_L$. Let $f \in L^2$ be given by

$$f(x) = \exp(-x^2) \qquad (x \in \mathbb{R}^n).$$

Then by the above we have $P_f \in \bar{\mathcal{A}}_L$. If $P_f \in \mathcal{A}_L$ then there is a bounded Borel set S with $P_f = E(S) P_f E(S)$. Let $g = \chi_{\mathbb{R}^n - S} f$ then $g \in L^2$ and $E(S) P_f E(S) g = 0$.

But $\quad P_f\, g = \dfrac{\langle f\,|\,g\rangle}{\|f\|^2} f = \dfrac{\|g\|^2}{\|f\|^2} f \neq 0$

so $P_f \neq E(S) P_f E(S)$ giving a contradiction. Hence $P_f \in \bar{\mathcal{A}}_L - \mathcal{A}_L$ so $\bar{\mathcal{A}}_L \neq \mathcal{A}_L$. ∎

We conclude this section with a discussion of the time evolution of local observables. In particular we show that if $A \in \bar{\mathcal{A}}_L$ then $U_t^* A U_t$ need not belong to $\bar{\mathcal{A}}_L$ where U is a free particle evolution group in $L^2(\mathbb{R}^n)$. It follows that the usual time evolution of a free particle cannot be described by a group of $*$-automorphisms of $\bar{\mathcal{A}}_L$.

(8.10) <u>Theorem</u>. Let U be an evolution group in L^2 then

$$\lim_{t\to\infty} \langle U_t f \,|\, A U_t g\rangle = 0$$

for all $A \in \bar{\mathcal{A}}_L$, all $f \in L^2$ and all scattering states g of U.

<u>Proof</u>. If $A \in \mathcal{A}_L$ then $A = E_Q(S) A E_Q(S)$ for some bounded Borel set S and since g is a scattering state of U

$$\begin{aligned} |\langle U_t f \,|\, A U_t g\rangle| &\leq \|f\| \, \|A U_t g\| \\ &= \|f\| \, \|E_Q(S) A E_Q(S) U_t g\| \\ &\leq \|f\| \, \|A\| \, \|E_Q(S) U_t g\| \\ &\to 0 \qquad (t \to \infty) . \end{aligned}$$

Now suppose $A \in \bar{\mathcal{A}}_L$ then there is a sequence A_r converging to A in the operator norm such that for each $r \in \mathbb{N}$

$$\lim_{t\to\infty} \langle U_t f \,|\, A_r U_t g\rangle = 0 .$$

Define functions u_r and u by

$$u_r(t) = \langle U_t f \,|\, A_r U_t g\rangle , \qquad u(t) = \langle U_t f \,|\, A U_t g\rangle$$

for $t \in \mathbb{R}$ then

$$\sup_{t\in\mathbb{R}} |u_r(t) - u(t)| = \sup_{t\in\mathbb{R}} |\langle U_t f | (A_r - A) U_t g\rangle |$$
$$\leq \|f\| \, \|A_r - A\| \, \|g\|$$
$$\rightarrow 0 \qquad (r \rightarrow \infty)$$

so u_r converges uniformly to u and hence (A1)

$$\lim_{t\to\infty} \langle U_t f | A U_t g\rangle = \lim_{r\to\infty} \lim_{t\to\infty} \langle U_t f | A_r U_t g\rangle = 0 .$$ ∎

(8.11) Theorem. Let U be the free particle evolution group in L^2 for a particle of mass m, then for all Borel sets S and all $t \neq 0$

$$U_t^* E_Q(S) U_t = \exp\left(\tfrac{-im}{2\hbar t} Q^2\right) E_P\left(\tfrac{m}{t} S\right) \exp\left(\tfrac{im}{2\hbar t} Q^2\right) .$$

Proof. Let $t \neq 0$ and let S be a Borel set. Let D_t and C_t° be the operators defined in (A3) by

$$(D_t f)(x) = |t|^{n/2} f(tx) \qquad (\forall f \in L^2)(\forall x \in \mathbb{R}^n)$$
$$C_t^\circ = a(t) D_{m/t} \exp\left(\tfrac{it}{2m\hbar} Q^2\right) F$$

where $a(t) = \exp(-in\pi t/4|t|)$ and F is the Fourier transform in L^2. For an arbitrary $f \in L^2$,

$$(D_{t/m} E_Q(S) D_{m/t} f)(x) = \left|\tfrac{t}{m}\right|^{n/2} \chi_S\left(\tfrac{t}{m}x\right) (D_{m/t} f)\left(\tfrac{t}{m}x\right)$$
$$= \chi_S\left(\tfrac{t}{m}x\right) f(x)$$
$$= \left(E_Q\left(\tfrac{m}{t}S\right) f\right)(x)$$

so $D_{m/t}^{-1} E_Q(S) D_{m/t} = E_Q\left(\frac{m}{t} S\right)$.

Now by (A3.4)

$$U_t = C_t^\circ \exp\left(\tfrac{im}{2\hbar t} Q^2\right) .$$

So using this identity, the above expression for $E_Q\left(\frac{m}{t}S\right)$ and the facts that $|a(t)|^2 = 1$ and $E_Q\left(\frac{m}{t}S\right)$ commutes with $\exp\left(\frac{im}{2\hbar t} Q^2\right)$ we have

$$U_t^* E_Q(S) U_t$$

$$= \exp\left(\frac{-im}{2\hbar t} Q^2\right) (C_t^o)^{-1} E_Q(S)\, C_t^o \exp\left(\frac{im}{2\hbar t} Q^2\right)$$

$$= \exp\left(\frac{-im}{2\hbar t} Q^2\right) F^{-1} \exp\left(\frac{-it}{2m\hbar} Q^2\right) D_{m/t}^{-1} E_Q(S)\, D_{m/t} \exp\left(\frac{it}{2m\hbar}\right) F \exp\left(\frac{im}{2\hbar t} Q^2\right)$$

$$= \exp\left(\frac{-im}{2\hbar t} Q^2\right) F^{-1} E_Q\left(\tfrac{m}{t} S\right) F \exp\left(\frac{im}{2\hbar t} Q^2\right)$$

$$= \exp\left(\frac{-im}{2\hbar t} Q^2\right) E_P\left(\tfrac{m}{t} S\right) \exp\left(\frac{im}{2\hbar t} Q^2\right). \quad \blacksquare$$

(8.12) Corollary. For all Borel sets R and S and all $t \neq 0$,

$$\| E_Q(R) U_t^* E_Q(S) U_t E_Q(R) - U_t^* E_Q(S) U_t \|$$

$$= \| E_Q(R) E_P\left(\tfrac{m}{t} S\right) E_Q(R) - E_P\left(\tfrac{m}{t} S\right) \|$$

Proof. Since $\exp\left(\frac{im}{2\hbar t} Q^2\right)$ is unitary and commutes with $E_Q(R)$ we have by the theorem,

$$\| E_Q(R) U_t^* E_Q(S) U_t E_Q(R) - U_t^* E_Q(S) U_t \|$$

$$= \left\| \exp\left(\frac{-im}{2\hbar t} Q^2\right) E_Q(R) E_P\left(\tfrac{m}{t} S\right) E_Q(R) \exp\left(\frac{im}{2\hbar t} Q^2\right) - \exp\left(\frac{-im}{2\hbar t} Q^2\right) E_P\left(\tfrac{m}{t} S\right) \exp\left(\frac{im}{2\hbar t} Q^2\right) \right\|$$

$$= \| E_Q(R) E_P\left(\tfrac{m}{t} S\right) E_Q(R) - E_P\left(\tfrac{m}{t} S\right) \| \quad \blacksquare$$

(8.13) Corollary. Let U be a free particle evolution group in L^2. If S is a bounded Borel set with $E_Q(S) \neq 0$ and if $t \neq 0$ then $U_t^* E_Q(S) U_t \notin \bar{\mathcal{A}}_L$.

Proof. Let $t \neq 0$ and suppose that $U_t^* E_Q(S) U_t$ does belong to $\bar{\mathcal{A}}_L$ then by (8.7) there is a sequence R_k of bounded Borel sets such that

$$\| E_Q(R_k) U_t^* E_Q(S) U_t E_Q(R_k) - U_t^* E_Q(S) U_t \| \longrightarrow 0 \qquad (k \to \infty)$$

and hence by (8.12)

$$\| E_Q(R_k) E_P\left(\tfrac{m}{t} S\right) E_Q(R_k) - E_P\left(\tfrac{m}{t} S\right) \| \longrightarrow 0 \qquad (k \to \infty)$$

which implies $E_P(\frac{m}{t}S) \in \mathcal{A}_L$.

Since every $f \in L^2$ is a scattering state of U (5.5) and U_t commutes with $E_P(\frac{m}{t}S)$ we now have from (8.10)

$$0 = \lim_{t\to\infty} \langle U_t f \mid E_P(\tfrac{m}{t}S) U_t f \rangle = \| E_P(\tfrac{m}{t}S) f \|^2.$$

This implies $E_P(\frac{m}{t}S) = 0$ and it follows that $E_Q(S) = 0$. ∎

9. ASYMPTOTIC OPERATOR ALGEBRAS.

In this section we study some algebras of operators in which quantum mechanical expectation values converge in time. We are particularly interested in constructing a C*-algebra on which spatially separating coherent states can evolve into disjoint states. Such algebras will then be used to tackle the paradoxes associated with spatial separation outlined in section 3.

Throughout this section $\mathcal{L}^2$ will denote the Hilbert space $\mathcal{L}^2(\mathbb{R}^n)$ for some fixed $n \in \mathbb{N}$ and E_Q and E_P will be the usual position and momentum spectral measures in $\mathcal{L}^2(\mathbb{R}^n)$. We fix a free evolution group U in $\mathcal{L}^2$. Recall from (5.5) that every vector in $\mathcal{L}^2$ is a scattering state of such an evolution group, and also from (2.7) that the parity operator commutes with U_t for every $t \in \mathbb{R}$.

(9.1) A C*-subalgebra $\mathcal{A}$ of $\mathcal{B}(\mathcal{L}^2)$ will be called an asymptotic algebra if

$$\lim_{t \to \infty} \langle U_t f \mid A U_t f \rangle$$

exists for all $A \in \mathcal{A}$ and all $f \in \mathcal{L}^2$.

Physically this convergence of expectation values enables us to define a state of the system as $t \to \infty$ in the following sense, (9.2) <u>Lemma</u>. Let $\mathscr{A}$ be an asymptotic algebra containing the identity operator and let f be a unit vector in L^2 . Then the formula

$$\omega(A) = \lim_{t\to\infty} \langle U_t f \mid A U_t f \rangle \qquad (A \in \mathscr{A})$$

defines a state ω on $\mathscr{A}$. If ω_t denotes the state defined by

$$\omega_t(A) = \langle U_t f \mid A U_t f \rangle \qquad (A \in \mathscr{A})$$

then $\omega_t \to \omega$ in the weak* topology on the dual space $\mathscr{A}^*$ of $\mathscr{A}$.

<u>Proof</u>. For every $A \in \mathscr{A}$

$$\omega(A^*A) = \lim_{t\to\infty} \| A U_t f \|^2 \geqslant 0$$

and

$$\omega(I) = \lim_{t\to\infty} \| U_t f \|^2 = 1 ,$$

so ω is a state. The convergence of ω_t to ω follows directly from the definition of the weak* topology. (For further details of this topology see Emch (1972a) p101.) ∎

Note that by (8.10) the C*-algebra $\bar{\mathscr{A}}_L$ is an asymptotic algebra, also $L^\infty(P)$ is clearly an asymptotic algebra. We begin our discussion of these algebras with a result which implies that $\mathbb{B}(L^2)$ is not an asymptotic algebra.

(9.3) <u>Theorem</u>. Let f be a non-zero positive scattering state of an evolution group V in L^2. Then there is a Borel set S for which

$$\lim_{t\to\infty} \langle V_t f \mid E_Q(S) V_t f \rangle$$

does not exist. The sets S and $\mathbb{R}^n - S$ are necessarily unbounded.

Proof. We may assume that $\|f\| = 1$. Let B_r denote a ball of radius r centred at the origin in $\mathbb{R}^n$.

Fix $\kappa \in \mathbb{N}$, $T > 0$ and $\varepsilon > 0$. Since f is a positive scattering state we have

$$\| E_Q(B_\kappa) V_t f \|^2 < \varepsilon/2$$

for some $t > T$. Also since the sequence $E_Q(B_r)^\perp$ converges strongly to 0 there is a $k > \kappa$ with

$$\| E_Q(B_k)^\perp V_t f \|^2 < \varepsilon/2 .$$

Now since $\mathbb{R}^n - (B_k - B_\kappa)$ is the union of the disjoint sets $\mathbb{R}^n - B_k$ and B_κ we have

$$\begin{aligned} \| E_Q(B_k - B_\kappa)^\perp V_t f \|^2 &= \| E_Q(B_k)^\perp V_t f \|^2 + \| E_Q(B_\kappa) V_t f \|^2 \\ &< \varepsilon . \end{aligned}$$

This establishes the following result,

$$\left.\begin{array}{l} \text{Given } \kappa,\ T \text{ and any } \varepsilon > 0 \text{ there is a } k > \kappa \\ \text{and a } t > T \text{ with} \\ \qquad \| E_Q(B_k - B_\kappa) V_t f \|^2 > 1 - \varepsilon . \end{array}\right\} \quad (*)$$

Now let $\kappa = T = \varepsilon = 1$ then by $(*)$ there is a $k_1 > 1$ and a $t_1 > 1$ with

$$\| E_Q(B_{k_1} - B_1) V_{t_1} f \|^2 > 1 - 1/1 = 0 .$$

Applying $(*)$ again with $\kappa = k_1$, $T = t_1 + 1$ and $\varepsilon = 1/2$ we deduce that there is a $k_2 > k_1$ and a $t_2 > t_1 + 1$ with

$$\| E_Q(B_{k_2} - B_{k_1}) V_{t_2} f \|^2 > 1 - 1/2 .$$

Continuing in this way we may use $(*)$ to obtain inductively, increasing sequences k_r and t_r both diverging to ∞ and having the property

$$\| E_Q(B_{k_r} - B_{k_{r-1}}) V_{t_r} f \|^2 > 1 - 1/r \qquad (r \in \mathbb{N}) .$$

Now define $S_r = B_{k_r} - B_{k_{r-1}}$ and let $S = S_2 \cup S_4 \cup \ldots$, then $\mathbb{R}^n - S = B_1 \cup S_1 \cup S_3 \cup \ldots$. For each $r \in \mathbb{N}$ we have

$$\| E_Q(S) V_{t_{2r}} f \|^2 \geq \| E_Q(S_{2r}) V_{t_{2r}} f \|^2 \geq 1 - \tfrac{1}{2r} \geq \tfrac{1}{2},$$

$$\| E_Q(S)^{\perp} V_{t_{2r+1}} f \|^2 \geq \| E_Q(S_{2r+1}) V_{t_{2r+1}} f \|^2 \geq 1 - \tfrac{1}{2r+1}$$

and the latter inequality gives

$$\| E_Q(S) V_{t_{2r+1}} f \|^2 \leq \tfrac{1}{2r+1}$$

But now we have $\lim_{r\to\infty} \| E_Q(S) V_{t_{2r+1}} f \|^2 = 0$ while $\| E_Q(S) V_{t_{2r}} f \|^2 \geq 1/2$ for every $r \in \mathbb{N}$ and it follows that $\lim_{t\to\infty} \langle V_t f | E_Q(S) V_t f \rangle$ does not exist.

Finally it is easily checked from the definition of a scattering state that the limit exists whenever S or $\mathbb{R}^n - S$ is bounded. ∎

Clearly any asymptotic algebra must be contained in the set

$$\{ A \in \mathbb{B}(L^2) : \lim_{t\to\infty} \langle U_t f | A U_t f \rangle \ \text{ exists } \ \forall f \in L^2 \},$$

so we now investigate some properties of this set. It will be useful at a later stage to consider sets of operators whose expectation values converge to zero. A useful description may be obtained by introducing the weak topology.

(9.4) <u>Definition.</u>

$$\mathcal{A}^W = \{ A \in \mathbb{B}(L^2) : \operatorname*{w-lim}_{t\to\infty} U_t^* A U_t \ \text{ exists} \},$$

$$\mathcal{A}_0^W = \{ A \in \mathbb{B}(L^2) : \operatorname*{w-lim}_{t\to\infty} U_t^* A U_t = 0 \}.$$

(9.5) <u>Lemma.</u> Let F be the Fourier transform in L^2 then

$$\lim_{t\to\infty} \langle U_t f | F U_t g \rangle = 0$$

for all $f, g \in L^2$.

Proof. We may assume that f and g are unit vectors. Let $\varepsilon > 0$ then there is a bounded Borel set S with $\|E_Q(S)^{\perp} Fg\| < \varepsilon/2$ and also since f is a scattering state of U we have $\|E_Q(S)U_t f\| < \varepsilon/2$ for all sufficiently large t. Since each U_t commutes with every $E_P(S)$ it follows that FU_tF^{-1} commutes with $E_Q(S)$ and hence for all sufficiently large t

$$\begin{aligned}
|\langle U_t f \mid FU_t g\rangle| &= |\langle U_t f \mid (E_Q(S) + E_Q(S)^{\perp})(FU_tF^{-1})Fg\rangle| \\
&\leqslant \|E_Q(S)U_t f\| + \|E_Q(S)^{\perp}(FU_tF^{-1})Fg\| \\
&= \|E_Q(S)U_t f\| + \|E_Q(S)^{\perp}Fg\| \\
&< \varepsilon
\end{aligned}$$

and the result follows. ∎

(9.6) Theorem.

[1] $\mathscr{A}^w = \{A \in \mathbb{B}(L^2): \lim_{t\to\infty} \langle U_t f \mid AU_t f\rangle$ exists for all $f \in L^2\}$;

[2] $\mathscr{A}_0^w = \{A \in \mathbb{B}(L^2): \lim_{t\to\infty} \langle U_t f \mid AU_t f\rangle = 0$ for all $f \in L^2\}$;

[3] $\mathscr{A}^w$ and $\mathscr{A}_0^w$ are not closed under multiplication.

Proof. [2] follows from the polarization identity for the sesquilinear form $(f,g) \mapsto \langle f \mid Ag\rangle$ on L^2 (Weidmann (1980) p2).

It is also clear from the polarization identity for this sesquilinear form that if $A \in \mathbb{B}(L^2)$ and $\lim_{t\to\infty}\langle U_t f \mid AU_t f\rangle$ exists then

$$s(f,g) = \lim_{t\to\infty} \langle U_t f \mid AU_t g\rangle$$

exists for all $f, g \in L^2$. Now this formula clearly defines a sesquilinear form s on L^2 which is bounded (by $\|A\|$) and it follows (Weidmann (1980) p120) that there is a unique $T \in \mathbb{B}(L^2)$ with $s(f,g) = \langle f \mid Tg\rangle$ for all f, g in L^2. Clearly $U_t^* A U_t$ converges weakly to T and [1] follows.

To prove [3] note that the Fourier transform F belongs to $\mathcal{A}_o^w$ by the lemma. If $\mathcal{A}_o^w$ is closed under multiplication then since $I = F^4$ (2.2) we have $I \in \mathcal{A}_o^w$ implying $\|f\|^2 = \lim_{t\to\infty} \langle U_t f | I U_t f\rangle = 0$ for all f in L^2, which is a contradiction. Hence $\mathcal{A}_o^w$ is not closed under multiplication and it only remains to show that $\mathcal{A}^w$ has the same property. Since U_t and $E_p(S)$ commute for every $t \in \mathbb{R}$ and every Borel set S it follows that each $E_p(S)$ belongs to $\mathcal{A}^w$. Now since $F \in \mathcal{A}^w$ the assumption that $\mathcal{A}^w$ is closed under multiplication gives

$$E_Q(S) = F^{-1} E_p(S) F = F^3 E_p(S) F \in \mathcal{A}^w.$$

Since this holds for all S we have contradicted the result of (9.3) and hence $\mathcal{A}^w$ is not closed under multiplication. ∎

We shall now investigate the set of operators A in $\mathbb{B}(L^2)$ posessing "asymptotically vanishing correlations" in the sense that $\lim_{t\to\infty} \langle U_t f | A U_t g\rangle = 0$ whenever f and g are spatially separating state vectors (7.1).

(9.7) Definition.

$$\mathcal{A}^w_{avc} = \{A \in \mathbb{B}(L^2):\ \underset{t\to\infty}{\text{w-lim}}\ E_p(S)^{\perp} U_t^* A U_t E_p(S) = 0 \quad \forall \text{ Borel sets } S\}.$$

(9.8) Corollary. Let $A \in \mathbb{B}(L^2)$ then these are equivalent,

[1] $A \in \mathcal{A}^w_{avc}$;

[2] $\underset{t\to\infty}{\text{w-lim}}\ E_p(R) U_t^* A U_t E_p(S) = 0$ for all disjoint Borel sets R and S;

[3] $\lim_{t\to\infty} \langle U_t f | A U_t g\rangle = 0$ whenever $f, g \in L^2$ are spatially separating with respect to U.

Proof. $([1] \Rightarrow [2])$ If $E_P(S)^{\perp} U_t^* A U_t E_P(S) \xrightarrow{w} 0$ then for all Borel sets R we have $E_P(R) E_P(S)^{\perp} U_t^* A U_t E_P(S) \xrightarrow{w} 0$. If R and S are disjoint then $R \cap (\mathbb{R}^n - S) = R$ so $E_P(R) E_P(S)^{\perp} = E_P(R)$ and [2] follows.

$([2] \Rightarrow [3])$ If f and g are spatially separating then by (7.3) there are disjoint Borel sets R and S with $E_P(R) f = f$ and $E_P(S) g = g$. So assuming [2]

$$\lim_{t\to\infty} \langle U_t f | A U_t g \rangle = \lim_{t\to\infty} \langle f \mid E_P(R) U_t^* A U_t E_P(S) g \rangle = 0 .$$

$([3] \Rightarrow [1])$ Let S be a Borel set and let $f, g \in \mathcal{H}$ then by (7.3) $E_P(S) g$ and $E_P(S)^{\perp} f = E_P(\mathbb{R}^n - S) f$ are spatially separating so assuming [3] gives

$$0 = \lim_{t\to\infty} \langle U_t E_P(S)^{\perp} f \mid A U_t E_P(S) g \rangle$$

and [1] follows. ∎

(9.9) Lemma. Let $\mathcal{P}$ be the parity operator in L^2, then there is a Borel set S in $\mathbb{R}^n$ and a unit vector f in L^2 with $E_P(S) f = f$

$$E_P(S)^{\perp} U_t^* \mathcal{P} U_t E_P(S) f = \mathcal{P} f.$$

Proof. Let $S = \{ (x_1, \ldots x_n) \in \mathbb{R}^n : x_i > 0,\ 1 \le i \le n \}$ and let $-S = \{ -x : x \in S \}$ then S and $-S$ are disjoint so $E_P(S)^{\perp} E_P(-S) = E_P(-S)$. It is easily verified that $\mathcal{P} E_Q(S) = E_Q(-S) \mathcal{P}$ and since $\mathcal{P} = F^2$ where F is the Fourier transform we have $\mathcal{P} E_P(S) = E_P(-S) \mathcal{P}$. Let f be a unit vector in the range of $E_P(S)$ then since $\mathcal{P}$ commutes with every U_t

$$\begin{aligned} E_P(S)^{\perp} U_t^* \mathcal{P} U_t E_P(S) f &= E_P(S)^{\perp} \mathcal{P} E_P(S) f \\ &= E_P(S)^{\perp} E_P(-S) \mathcal{P} f \\ &= E_P(-S) \mathcal{P} f \end{aligned}$$

$$= \quad \wp E_p(S) f$$

$$= \quad \wp f \qquad \blacksquare$$

(9.10) Corollary. The parity operator $\wp$ in L^2 does not belong to $\mathcal{A}^w_{avc}$.

Proof. If f is a unit vector having the property described in the lemma then since $\wp$ is unitary

$$\lim_{t\to\infty} \langle \wp f \mid E_p(S)^{\perp} U_t^* \wp U_t E_p(S) f \rangle \;=\; \langle \wp f \mid \wp f \rangle \;=\; 1$$

so $\wp$ does not belong to $\mathcal{A}^w_{avc}$.

The relationship between the sets $\mathcal{A}^w$, $\mathcal{A}^w_o$ and $\mathcal{A}^w_{avc}$ is the subject of the next theorem. $\bar{\mathcal{A}}_L$ will denote the C*-algebra generated by local observables as defined in (8.1).

(9.11) Theorem.

[1] $\bar{\mathcal{A}}_L \subset \mathcal{A}^w_o \subset \mathcal{A}^w_{avc} \subset \mathbb{B}(\mathcal{H})$,

[2] $\mathcal{A}^w_{avc} \not\subseteq \mathcal{A}^w$,

[3] $\mathcal{A}^w_{avc} \cap \mathcal{A}^w = \{A \in \mathbb{B}(L^2) : \underset{t\to\infty}{\text{w-lim}}\ U_t^* A U_t \text{ exists and belongs to } L^\infty(P)\}$

$$= \mathcal{A}^w_o + L^\infty(P),$$

[4] Neither of the sets $\mathcal{A}^w_{avc}$ or $\mathcal{A}^w_{avc} \cap \mathcal{A}^w$ is closed under multiplication.

Proof. [1]: $\bar{\mathcal{A}}_L \subseteq \mathcal{A}^w_o$ by (8.10) and $\mathcal{A}^w_o \subseteq \mathcal{A}^w_{avc}$ is easily verified. The second inclusion is proper since $I \in \mathcal{A}^w_{avc} - \mathcal{A}^w_o$ follows immediately from the definitions, and the last inclusion is proper since the parity operator does not belong to $\mathcal{A}^w_{avc}$ by the last corollary.

To prove the first inclusion is also proper we first assume U is the free particle evolution group for a particle of mass m. Fix $s > 0$ and let R be a bounded Borel set then since $E_Q(R) \in \mathcal{A}_0^w$

$$\operatorname*{w-lim}_{t\to\infty} U_t^*(U_s^* E_Q(R) U_s) U_t \;=\; \operatorname*{w-lim}_{t\to\infty} U_t^* E_Q(R) U_t \;=\; 0$$

so $U_s^* E_Q(R) U_s \in \mathcal{A}_w^0$. But by (8.13) we can choose R such that $U_s^* E_Q(R) U_s \notin \bar{\mathcal{A}}_L$ and hence in this case the inclusion is proper. Now suppose U is a free evolution group for a system of two particles then n must be even and the evolution group U may be written in the form $U_t = U_t^1 \otimes U_t^2$ where U^1 and U^2 are free particle evolution groups in $L^2(\mathbb{R}^r)$ where $r = n/2$. If $f, g \in L^2(\mathbb{R}^r)$ then for any bounded Borel set S in $\mathbb{R}^r$

$$\begin{aligned}\lim_{t\to\infty} \| U_t^* E_Q(S\times\mathbb{R}^r) U_t (f\otimes g)\| &\leq \lim_{t\to\infty} \| E(S) U_t^1 f\| \, \|g\| \\ &= 0\end{aligned}$$

where E is the position spectral measure in $L^2(\mathbb{R}^r)$. It follows that $\lim_{t\to\infty} U_t^* E_Q(S\times\mathbb{R}^r) U_t h = 0$ for all h in some dense linear manifold in $L^2(\mathbb{R}^n)$. Now suppose $h \in L^2(\mathbb{R}^n)$ is arbitrary and choose a sequence h_k converging to h such that

$$\lim_{t\to\infty} U_t^* E_Q(S\times\mathbb{R}^r) U_t h_k = 0 \qquad (\forall k \in \mathbb{N}).$$

Define u_k and u by

$$u_k(t) = U_t^* E_Q(S\times\mathbb{R}^r) U_t h_k \;, \qquad u(t) = U_t^* E_Q(S\times\mathbb{R}^r) U_t h$$

for all $t \in \mathbb{R}$, then

$$\sup_{t\in\mathbb{R}} \| u_k(t) - u(t)\| \leq \| h_k - h \|$$

so u_k converges uniformly to u and hence

$$\lim_{t\to\infty} U_t^* E_Q(S\times\mathbb{R}^r) U_t h = \lim_{k\to\infty}\lim_{t\to\infty} U_t^* E_Q(S\times\mathbb{R}^r) U_t h_k = 0 .$$

It follows that $E_Q(S\times\mathbb{R}^r) \in \mathcal{A}_0^w$. By way of contradiction suppose $E_Q(S\times\mathbb{R}^r) \in \bar{\mathcal{A}}_L$ then by (8.7) $E_Q(B_k \cap (S\times\mathbb{R}^r)) \to E_Q(S\times\mathbb{R}^r)$ in the operator norm for some sequence B_k of bounded Borel sets in $\mathbb{R}^n$. Since this sequence of projections is not eventually constant it cannot converge in the operator norm (c.f. the remark at the end of (8.8)) so we have arrived at the required contradiction. Hence $E_Q(S\times \mathbb{R}^r) \in \mathcal{A}_0^w - \tilde{\mathcal{A}}_L$ and the first inclusion is proper.

[2]: We have just shown in (9.10) that the parity operator $\mathcal{P}$ does not belong to $\mathcal{A}_{avc}^w$ and since $\mathcal{P}$ commutes with U_t for every t, $\mathcal{P}$ belongs to $\mathcal{A}^w$.

[3]: Let $\mathcal{A} = \{A\in\mathbb{B}(L^2):\ \underset{t\to\infty}{\text{w-lim}}\ U_t^* A U_t$ exists and belongs to $L^\infty(P)\}$ and for every $t\in\mathbb{R}$ and every $A\in\mathbb{B}(L^2)$ let $A_t = U_t^* A U_t$. We show

$$\mathcal{A}_{avc}^w \cap \mathcal{A}^w \subseteq \mathcal{A} \subseteq \mathcal{A}_0^w + L^\infty(P) \subseteq \mathcal{A}_{avc}^w \cap \mathcal{A}^w.$$

Suppose $A\in \mathcal{A}_{avc}^w \cap \mathcal{A}^w$ and let $T = \underset{t\to\infty}{\text{w-lim}}\ A_t$. Since $A\in\mathcal{A}_{avc}^w$ we have for any Borel set S

$$E_P(S)^\perp T E_P(S) = \underset{t\to\infty}{\text{w-lim}}\ E_P(S)^\perp A_t E_P(S) = 0,$$

which implies $E_P(S)^\perp T E_P(S) = 0 = E_P(S) T E_P(S)^\perp \quad (\forall S).$
Therefore for any S,

$$\begin{aligned} TE_P(S) &= \left(E_P(S)^\perp + E_P(S)\right) T E_P(S) \\ &= E_P(S)\ T E_P(S) \\ &= E_P(S)\ T\ (E_P(S) + E_P(S)^\perp) \\ &= E_P(S)\, T \end{aligned}$$

and it follows that $T\in L^\infty(P)' = L^\infty(P)$ (see (2.3)). Hence $\mathcal{A}_{avc}^w \cap \mathcal{A}^w \subseteq \mathcal{A}$.

Now let $A\in\mathcal{A}$ and let $T = \underset{t\to\infty}{\text{w-lim}}\ A_t$ then $T\in L^\infty(P)$ so T commutes with every U_t and for all f and g we have

$$|\langle f\,|\,(A-T)_t\, g\rangle| \le |\langle f\,|\,(A_t - T) g\rangle| \to 0$$

so $A - T\in\mathcal{A}_0^w$ which implies $A\in\mathcal{A}_0^w + L^\infty(P)$.

Finally suppose $A\in\mathcal{A}_0^w + L^\infty(P)$ then $A = B + T$ for some

$B \in \mathscr{A}_0^w$ and some $T \in L^\infty(P)$. It follows directly from the definitions ((9.4),(9.7)) that $B, T \in \mathscr{A}^w \cap \mathscr{A}_{avc}^w$ (since T commutes with every U_t and every $E_P(S)$) and also that $B + T \in \mathscr{A}^w \cap \mathscr{A}_{avc}^w$.

[4]: By (9.5) the Fourier transform F belongs to $\mathscr{A}_0^w$ and hence to $\mathscr{A}_{avc}^w$ and to $\mathscr{A}_{avc}^w \cap \mathscr{A}^w$. Now $F^2 = \mathscr{P}$ the parity operator which does not belong to $\mathscr{A}_{avc}^w$ by (9.10) and so does not belong to $\mathscr{A}_{avc}^w \cap \mathscr{A}^w$ either. Thus neither of these sets is closed under multiplication. ∎

(9.12) We now investigate the consequences of altering the topology in the definitions of the above sets. If τ is a topology on $\mathbb{B}(L^2)$ which is compatible with the vector space structure then

$$\mathscr{A}^\tau = \{ A \in \mathbb{B}(L^2) : \tau\text{-}\lim_{t \to \infty} U_t^* A U_t \text{ exists} \}$$

will be a linear manifold in $\mathbb{B}(L^2)$. If τ is finer than the weak topology then $\mathscr{A}^\tau \subseteq \mathscr{A}^w$, so if $\mathscr{A}^\tau$ is an algebra we can define a limit of states on $\mathscr{A}^\tau$ as in (9.2). The simplest case is where τ is the uniform (or norm topology).

(9.13) <u>Definition.</u> We define sets of operators $\mathscr{A}^u$, $\mathscr{A}_0^u$ and $\mathscr{A}_{avc}^u$ in $\mathbb{B}(L^2)$ by

$$\mathscr{A}^u = \{A \in \mathbb{B}(L^2): \underset{t \to \infty}{\text{u-lim}}\ U_t^* A U_t \text{ exists}\},$$

$$\mathscr{A}_0^u = \{A \in \mathbb{B}(L^2): \underset{t \to \infty}{\text{u-lim}}\ U_t^* A U_t = 0\},$$

$$\mathscr{A}_{avc}^u = \{A \in \mathbb{B}(L^2): \underset{t \to \infty}{\text{u-lim}}\ E_p(S)^\perp U_t^* A U_t E_p(S) = 0 \ \ \forall \text{Borel sets } S \}$$

(9.14) <u>Theorem.</u>

[1] $\mathcal{A}_o^u = \{0\}$, $\mathcal{A}^u = \{H_o\}'$ and $\mathcal{A}_{avc}^u = L^\infty(P)$,

where H_o is the generator of the evolution group U.

[2] $\mathcal{A}^u$ and $\mathcal{A}_{avc}^u$ are von Neumann algebras on L^2 and

$$\mathcal{A}_o^u \subset \mathcal{A}_{avc}^u \subset \mathcal{A}^u \subset \mathcal{B}(L^2).$$

<u>Proof.</u> For each $A \in \mathcal{B}(L^2)$ and each $t \in \mathbb{R}$ let $A_t = U_t^* A U_t$.

[1]: If $A \in \mathcal{A}_o^u$ then since $\|A\| = \|U_t^* A U_t\|$ for each $t \in \mathbb{R}$, letting $t \to \infty$ gives $\|A\| = 0$. Hence $\mathcal{A}_o^u = \{0\}$.

If A commutes with H_o then A commutes with each U_t (see (A4)) so $\{H_o\}' \subseteq \mathcal{A}^u$. Conversely suppose $A \in \mathcal{A}^u$ and let $L = \underset{t\to\infty}{u\text{-}\lim}\, A_t$ then for all s in $\mathbb{R}$

$$\begin{aligned} \| L - L_s \| &\leq \| L - A_{s+t} \| + \| A_{s+t} - L_s \| \\ &= \| L - A_{s+t} \| + \| A_t - L \| \end{aligned}$$

and letting $t \to \infty$ gives $L = L_s$. Hence L commutes with every U_s. Now let E be a spectral projection of H_o then E commutes with L (see (A4)) and E commutes with U_t so

$$\begin{aligned} \| AE - EA \| &= \| U_t^*(AE - EA) U_t \| \\ &= \| A_t E - E A_t \| \\ &\to \| LE - EL \| = 0 \qquad (t \to \infty) \end{aligned}$$

and it follows that A commutes with H_o so $\mathcal{A}^u \subseteq \{H_o\}'$.

We now prove the final equality in [1].

$$\begin{aligned} A \in L^\infty(P) &\Rightarrow (\forall S) \quad \| E_P(S)^\perp A_t E_P(S) \| = \| E_P(S)^\perp E_P(S) A \| = 0 \\ &\Rightarrow A \in \mathcal{A}_{avc}^u . \end{aligned}$$

$$A \in \mathcal{A}^u_{avc} \Rightarrow (\forall S) \quad \lim_{t\to\infty} \| E_p(S)^\perp A_t E_p(S) \| = 0$$
$$\Rightarrow (\forall S) \quad \| E_p(S)^\perp A E_p(S) \| = 0$$
$$\Rightarrow (\forall S) \quad A E_p(S) = E_p(S) A E_p(S) = E_p(S) A$$
$\Rightarrow$ A commutes with every element of $L^\infty(P)$
$$\Rightarrow A \in L^\infty(P) \qquad \text{(see (2.3))}.$$

[2] Most of the assertions in [2] follow immediately from [1]. The inclusion $L^\infty(P) \subset \{H_0\}'$ is proper since the parity operator commutes with H_0 and does not belong to $L^\infty(P)$. ∎

Hence by considering limits in the uniform topology we obtain von Neumann algebras $\mathcal{A}^u$ and $\mathcal{A}^u_{avc}$. There are a number of reasons why these algebras will not be suitable for describing a quantum mechanical system. The parity operator $\mathcal{P}$ commutes with a free particle Hamiltonian H_0 so by (9.10) there are operators in $\{H_0\}'$ which do not produce asymptotically vanishing correlations between spatially separating states. Also neither of these algebras contains any non-zero spectral projections of position so there is no way of describing the localisation of the system.

(9.15) We now investigate convergence in the strong and strong* topologies. Since the mapping $A \mapsto A^*$ is not continuous in the strong topology it will be convenient to introduce the strong* topology (Bratteli and Robinson (1979) p69) in which the adjoint is a continuous mapping. If A_r is a net in $\mathcal{B}(L^2)$ then we shall write $\text{s*-}\lim_{r\to\infty} A_r = A$ to mean that $\text{s-}\lim_{r\to\infty} A_r = A$ and $\text{s-}\lim_{r\to\infty} A_r^* = A^*$. To justify introducing strong* convergence we first show that taking the topology τ in (9.12) to be the strong topology does not make the set $\mathcal{A}^\tau$ a *-algebra.

(9.16) <u>Theorem.</u> The set $\{A \in \mathcal{B}(L^2) : \operatorname*{s-lim}_{t\to\infty} U_t^* A U_t \text{ exists}\}$ is not closed under the adjoint operation.

<u>Proof.</u> Let S be a bounded Borel set in $\mathbb{R}^n$ and let F be the Fourier transform in L^2. Then

$$\lim_{t\to\infty} \| U_t^* F^{-1} E_Q(S) U_t f \| = \lim_{t\to\infty} \| E_Q(S) U_t f \| = 0$$

for all $f \in L^2$ since each f is a scattering state of U. Hence the above set contains the operator $A = F^{-1}E_Q(S)$. Now $U_t^* A U_t \xrightarrow{s} 0$ and it follows that $U_t^* A^* U_t \xrightarrow{w} 0$ so if $\operatorname*{s-lim}_{t\to\infty} U_t^* A^* U_t$ exists it must also equal 0. But

$$\begin{aligned} \| U_t^* A U_t f \| &= \| E_Q(S) F U_t f \| \\ &= \| F^{-1} E_Q(S) F U_t f \| \\ &= \| E_P(S) U_t f \| \\ &= \| E_P(S) f \| \end{aligned}$$

which is non-zero for some S and f. Hence $\operatorname*{s-lim}_{t\to\infty} U_t^* A^* U_t$ does not exist. ∎

(9.17) <u>Definition.</u>

$$\mathcal{A}^s = \{A \in \mathcal{B}(L^2) : \operatorname*{s^*-lim}_{t\to\infty} U_t^* A U_t \text{ exists}\}$$

$$\mathcal{A}_0^s = \{A \in \mathcal{B}(L^2) : \operatorname*{s^*-lim}_{t\to\infty} U_t^* A U_t = 0\}$$

$$\mathcal{A}_{avc}^s = \{A \in \mathcal{B}(L^2) : \operatorname*{s^*-lim}_{t\to\infty} E_p(S)^\perp U_t^* A U_t E_p(S) = 0 \quad \forall \text{ Borel sets } S\}$$

(9.18) <u>Theorem.</u> $\mathcal{A}_0^s$ and $\mathcal{A}^s$ are proper C*-subalgebras of $\mathcal{B}(L^2)$ and $\mathcal{A}_0^s$ is a closed ideal in $\mathcal{A}^s$. The parity operator belongs to $\mathcal{A}^s$, but not to $\mathcal{A}_{avc}^s$.

Proof. It is easily verified that $\mathcal{A}_0^s$ and $\mathcal{A}^s$ are self-adjoint linear manifolds in $\mathcal{B}(L^2)$ and since the parity operator commutes with U_t for every $t \in \mathbb{R}$ it belongs to $\mathcal{A}^s$. It follows from (9.10) that $\mathcal{P} \notin \mathcal{A}^s_{ave}$. It is now sufficient to prove the following,

[1] $\mathcal{A}_0^s$ and $\mathcal{A}^s$ are closed in the operator norm,

[2] $A, B \in \mathcal{A}^s \Rightarrow AB \in \mathcal{A}^s$,

[3] $A \in \mathcal{A}_0^s,\ B \in \mathcal{A}^s \Rightarrow AB,\ BA \in \mathcal{A}_0^s$.

We begin by proving [1]. Let A_r be a sequence in $\mathcal{A}^s$ converging in the operator norm to A, and for each $r \in \mathbb{N}$ let $T_r = \operatorname*{s-lim}_{t\to\infty} U_t^* A_r U_t$. Now for any f in L^2

$$\begin{aligned} \| (T_r - T_s) f \| &= \lim_{t\to\infty} \| U_t^* (A_r - A_s) U_t f \| \\ &\leq \| f \| \, \| A_r - A_s \| \end{aligned}$$

so the sequence T_r is strongly Cauchy and hence converges strongly to some operator T say (Weidmann (1980) p75). Now fix a vector f and define u_r and u by

$$u_r(t) = U_t^* A_r U_t f - T_r f, \qquad u(t) = U_t^* A U_t f - Tf$$

for all $t \in \mathbb{R}$, then

$$\begin{aligned} \sup_{t\in\mathbb{R}} \| u_r(t) - u(t) \| &\leq \| (A_r - A) U_t f \| + \| (T - T_r) f \| \\ &\leq \| A_r - A \| \, \| f \| + \| (T - T_r) f \| \\ &\longrightarrow 0 \qquad (r \to \infty) \end{aligned}$$

since $A_r \longrightarrow A$ and $T_r \xrightarrow{s} T$. Hence the sequence u_r converges uniformly to u so (A1)

$$\lim_{t\to\infty} (U_t^* A U_t - T) f = \lim_{r\to\infty} \lim_{t\to\infty} (U_t^* A_r U_t - T_r) f = 0$$

Since f was arbitrary we have $\operatorname*{s-lim}_{t\to\infty} U_t^* A U_t = T$. Also since A_r^* converges in the operator norm to A^* it follows from an identical argument that $\operatorname*{s-lim}_{t\to\infty} U_t^* A^* U_t$ exists and hence (by considering the corrsponding weak limits) the value of this limit must be T^*. We have now shown $T = \underset{t\to\infty}{s^*\text{-}\lim}\, U_t^* A U_t$. Note that if the sequence A_r lies in $\mathcal{A}_0^s$ then each T_r equals 0 so $T=0$. Hence $\mathcal{A}_0^s$ and $\mathcal{A}^s$ are closed as required.

We now prove [2] and [3]. For each $A \in \mathbb{B}(L^2)$ and each $t \in \mathbb{R}$ let $A_t = U_t^* A U_t$. Now fix $A, B \in \mathcal{A}^s$ and let $X = \underset{t\to\infty}{s\text{-}\lim}\, A_t$, $Y = \underset{t\to\infty}{s\text{-}\lim}\, B_t$ then the inequalities

$$\|(AB)_t f - XYf\| \leq \|(A_t B_t - A_t Y)f\| + \|(A_t Y - XY)f\|$$
$$\leq \|A\|\,\|(B_t - Y)f\| + \|(A_t - X)Yf\|$$

imply $\underset{t\to\infty}{s\text{-}\lim}\,(AB)_t = XY$. Both [2] and [3] follow from this. ∎

Note that since strong* convergence implies weak convergence, $\mathcal{A}^s$ is an asymptotic algebra and hence any C*-subalgebra of $\mathcal{A}^s$ is also an asymptotic algebra. However since the parity operator belongs to $\mathcal{A}^s$ we do not have $\langle U_t f | A U_t g\rangle \to 0$ for all spatially separating pairs of vectors f and g. These correlations do however converge to 0 for operators in $\mathcal{A}^s \cap \mathcal{A}^s_{avc}$ and we now show that this set is a C*-algebra and obtain some alternative ways of expressing it.

(9.19) <u>Lemma.</u> Let $A \in \mathbb{B}(L^2)$ then $A \in \mathcal{A}^s_{avc}$ if and only if $E_p(S)^{\perp} A E_p(S) \in \mathcal{A}_0^s$ for evry Borel set S.

<u>Proof.</u> This follows immediately from the fact that U_t commutes with $E_p(S)$ for every t and S. ∎

(9.20) Theorem.

[1] $\mathcal{A}^s \cap \mathcal{A}^s_{avc}$ is a C*-subalgebra of $\mathbb{B}(L^2)$ and is equal to each of the following sets: $\mathcal{A}^s_o + L^\infty(P)$,

$\{A \in \mathbb{B}(L^2)$: $s^*\text{-}\lim_{t\to\infty} U_t^* A U_t$ exists and belongs to $L^\infty(P)\}$.

[2] $\bar{\mathcal{A}}_L \subset \mathcal{A}^s_o \subset \mathcal{A}^s \cap \mathcal{A}^s_{avc} \subset \mathcal{A}^s \subset \mathbb{B}(L^2)$.

[3] $\mathcal{A}^s_o$ is a closed ideal in $\mathcal{A}^s \cap \mathcal{A}^s_{avc}$.

Proof. [1] Let $\mathcal{A}$ denote the set

$\{A \in \mathbb{B}(L^2)$: $s^*\text{-}\lim_{t\to\infty} U_t^* A U_t$ exists and belongs to $L^\infty(P)\}$.

Then it follows from the inequalities at the end of the proof of (9.18) that $\mathcal{A}$ is closed under multiplication, and now clearly $\mathcal{A}$ is a *-subalgebra.

We now show $\mathcal{A}^s_{avc}$ is closed in the operator norm. Let A_r be a sequence in $\mathcal{A}^s_{avc}$ converging in the operator norm to $A \in \mathbb{B}(L^2)$, and let S be a fixed Borel set. Then the sequence $E(S)^\perp A_r E(S)$ converges to $E(S)^\perp A E(S)$ in the operator norm. Now by the lemma each $E(S)^\perp A_r E(S)$ belongs to $\mathcal{A}^s_o$ and since $\mathcal{A}^s_o$ is closed we have $E(S)^\perp A E(S) \in \mathcal{A}^s_o$ and it follows that $\mathcal{A}^s_{avc}$ is closed. Now by (9.18) $\mathcal{A}^s \cap \mathcal{A}^s_{avc}$ is an intersection of closed sets and hence closed.

To complete the proof of [1] it is now sufficient to prove the three sets are equal which we do by verifying the following inclusions,

$$\mathcal{A}^s \cap \mathcal{A}^s_{avc} \subseteq \mathcal{A} \subseteq \mathcal{A}^s_o + L^\infty(P) \subseteq \mathcal{A}^s \cap \mathcal{A}^s_{avc}$$

For every $t \in \mathbb{R}$ and every $A \in \mathbb{B}(L^2)$ let $A_t = U_t^* A U_t$.

Firstly let $A \in \mathcal{A}$ and let $T = s^*\text{-}\lim_{t\to\infty} A_t$ then since $A \in \mathcal{A}^s_{avc}$ we have for any Borel set S

$$E_P(S)^{\perp} T E_P(S) = \underset{t\to\infty}{s\text{-}\lim} E_P(S)^{\perp} A_t E_P(S) = 0$$

which implies

$$E_P(S)^{\perp} T E_P(S) = 0 = E_P(S) T E_P(S)^{\perp}$$

for all S. It follows as in the proof of (9.11) part [3] that $T \in L^{\infty}(P)$. Hence $\mathcal{A}^s \cap \mathcal{A}^s_{avc} \subseteq \mathcal{A}$.

Now let $A \in \mathcal{A}$ and let $T = \underset{t\to\infty}{s^*\text{-}\lim} A_t$ then $T \in L^{\infty}(P)$ so T commutes with every U_t and for all f we have

$$\| (A-T)_t f \| + \| (A-T)^*_t f \| = \| (A_t - T) f \| + \| (A_t - T)^* f \|$$

$$\to 0 \qquad (t \to \infty)$$

Hence $A - T \in \mathcal{A}^s_0$ which implies $A \in \mathcal{A}^s_0 + L^{\infty}(P)$.

Finally suppose $A \in \mathcal{A}^s_0 + L^{\infty}(P)$ then $A = B + T$ for some $B \in \mathcal{A}^s_0$ and some $T \in L^{\infty}(P)$. It follows directly from the definitions (9.17) that $B + T \in \mathcal{A}^s \cap \mathcal{A}^s_{avc}$ (since T commutes with every U_t and every $E_P(S)$). Hence $A \in \mathcal{A}^s \cap \mathcal{A}^s_{avc}$ and this completes the proof of [1].

[2] Let $A \in \mathcal{A}_L$ then $A = E_Q(S) A E_Q(S)$ for some bounded Borel set S. Let $f \in L^2$ then f is a scattering state of U so

$$\lim_{t\to\infty} \| U_t^* A U_t f \| \leq \| A \| \lim_{t\to\infty} \| E_Q(S) U_t f \|$$

$$= 0$$

and similarly $\| U_t^* A^* U_t f \| \to 0$ so $A \in \mathcal{A}^s_0$. Since $\mathcal{A}^s_0$ is closed (9.18) it follows that $\bar{\mathcal{A}}_L \subseteq \mathcal{A}^s_0$. The second inclusion in [2] is obvious from [1], the next follows from (9.18) and the last from (9.3) since (as we have already noted) $\mathcal{A}^s$ is an asymptotic algebra.

To show $\bar{\mathcal{A}}_L \subset \mathcal{A}^s_0$ first suppose U is the free particle evolution group of a particle of mass m. Let R be a bounded Borel set then for a fixed s

$$\underset{t\to\infty}{s^*\text{-}\lim}\ U_t^*(U_s^* E_Q(R) U_s) U_t = \underset{t\to\infty}{s\text{-}\lim}\ U_t^* E_Q(R) U_t$$

$$= 0$$

so $U_s^* E_Q(R) U_s \in \mathcal{A}_0^S$. But by (8.13) we can choose R and s such that $U_s^* E_Q(R) U_s \notin \bar{\mathcal{A}}_L$. Now if U is the free particle evolution group of a system of two particles then it follows from the proof of (9.11) part [1] that

$$\underset{t\to\infty}{s\text{-}\lim}\ U_t^* E_Q(S) U_t = 0$$

for some Borel set S with $E_Q(S) \notin \bar{\mathcal{A}}_L$. This is in fact a strong* limit since the terms are Hermitian and it follows that $E_Q(S) \in \mathcal{A}_0^S - \bar{\mathcal{A}}_L$.

Finally [3] follows from the facts that $\mathcal{A}_0^S \subseteq \mathcal{A}^S \cap \mathcal{A}_{avc}^S$ and $\mathcal{A}_0^S$ is a closed ideal in $\mathcal{A}^S$ (9.18). ∎

(9.21) Theorem. Let V be an evolution group in L^2 and assume that the wave operator

$$W_+ = \underset{t\to\infty}{s\text{-}\lim}\ V_t^* U_t$$

exists and is complete. Then $V_t \in \mathcal{A}_0^S + L^\infty(P)$ for every $t \in \mathbb{R}$.

Proof. We recall that the subspace of all scattering states of V is invariant under each V_t (5.3) and it follows that $E_\infty(V)$ commutes with V_t for every $t \in \mathbb{R}$. Now fix $s \in \mathbb{R}$ then for every f in L^2

$$\| U_t^* V_s E_\infty(V)^\perp U_t f \| = \| E_\infty(V)^\perp U_t f \|$$

$$= \| E_\infty(V)^\perp V_t^* U_t f \|$$

$$\to \| E_\infty(V)^\perp W_+ f \| \qquad (t \to \infty)$$

$$= 0$$

since $E_\infty(V)$ is the final domain of W_+ (5.7). Also

$$\| U_t^* E_\infty(V)^\perp V_s^* U_t f \| \quad = \quad \| E_\infty(V) U_t f \|$$

$$= \quad \| E_\infty(V)^\perp V_t^* U_t f \| \quad \longrightarrow \quad 0 \qquad (t \to \infty)$$

so $\quad \underset{t \to \infty}{s^*\text{-}\lim} \; U_t^* \left(V_s E_\infty(V)^\perp \right) U_t \;=\; 0 \, .$

Also since $\quad W_+^* \;=\; \underset{t \to \infty}{s\text{-}\lim} \; U_t^* V_t E_\infty(V)$

(5.7), it follows that

$$\begin{aligned} U_t^* V_s E_\infty(V) U_t \quad &= \quad \left(U_t^* V_t E_\infty(V) \right) V_s \left(V_t^* U_t \right) \\ &\xrightarrow{s} \quad W_+^* V_s W_+ \qquad\qquad (t \to \infty) \\ &= \quad W_+^* W_+ U_s \\ &= \quad U_s \end{aligned}$$

where we have used a multiplicative property of strong convergence and some results stated in (5.7). Similarly

$$\begin{aligned} U_t^* E_\infty(V) V_s^* U_t \quad &= \quad \left(U_t^* V_t E_\infty(V) \right) V_s^* \left(V_t^* U_t \right) \\ &\xrightarrow{s} \quad W_+^* V_s^* W_+ \qquad\qquad (t \to \infty) \\ &= \quad U_s^* \end{aligned}$$

so $\quad \underset{t \to \infty}{s^*\text{-}\lim} \; U_t^* V_s E_\infty(V) U_t \;=\; U_s$

Now we have

$$\begin{aligned} \underset{t \to \infty}{s^*\text{-}\lim} \; U_t^* V_s U_t \quad &= \quad \underset{t \to \infty}{s^*\text{-}\lim} \; U_t^* V_s E_\infty(V)^\perp U_t \;+\; \underset{t \to \infty}{s^*\text{-}\lim} \; U_t^* V_s E_\infty(V) U_t \\ &= \quad 0 \;+\; U_s \\ &\in \quad \mathcal{L}^\infty(P) \end{aligned}$$

and so $\quad V_s \in {}_{\sim}\!\mathcal{A}_0^s + \mathcal{L}^\infty(P)$ by (9.20). ∎

In particular the existence and completeness of W_+ implies that for each $t \in \mathbb{R}$ the mapping $\alpha_t : {}_{\sim}\!\mathcal{A}_0^s + \mathcal{L}^\infty(P) \longrightarrow {}_{\sim}\!\mathcal{A}_0^s + \mathcal{L}^\infty(P)$ defined by $\alpha_t(A) = V_t^* A V_t$ is a $*$-automorphism of ${}_{\sim}\!\mathcal{A}_0^s + \mathcal{L}^\infty(P)$. It is easily verified that this defines a continuous homomorphism from the group $\mathbb{R}$ to the group of $*$-automorphisms of ${}_{\sim}\!\mathcal{A}_0^s + \mathcal{L}^\infty(P)$ where the latter is equipped with the strong operator topology.

We shall show in the next chapter that the C*-algebra $\mathscr{A}_0^s + L^\infty(P)$ may be used to give an algebraic formulation of quantum mechanics in which the paradoxes described in section 3 can be resolved asymptotically. Note that by (9.20) every element of this algebra posesses asymptotically vanishing correlations. Also (9.21) implies that if the system is approximately free at large times then its evolution group induces a one-parameter family of *-automorphisms of $\mathscr{A}_0^s + L^\infty(P)$.

CHAPTER IV. ASYMPTOTICALLY SEPARABLE QUANTUM MECHANICS.

10. STATES ON ASYMPTOTIC ALGEBRAS AND THE DE BROGLIE PARADOX.

Throughout this section L^2 will denote the Hilbert space $L^2(\mathbb{R}^n)$ for some fixed $n \in \mathbb{N}$. We fix a free evolution group U in L^2 and let $\mathcal{A}_0^s$, $\mathcal{A}^s$ and $\mathcal{A}_{avc}^s$ be the sets defined in (9.17). By (9.20) the following C*-algebras are all equal to each other

$$\mathcal{A}^s \cap \mathcal{A}_{avc}^s, \quad \mathcal{A}_0^s + L^\infty(P)$$

and $\{A \in \mathbb{B}(L^2): \underset{t\to\infty}{s^*\text{-lim}}\, U_t^* A U_t$ exists and belongs to $L^\infty(P)\}$

and we shall throughout this section denote each of them by the symbol $\mathcal{A}$.

We now discuss the convergence of states on the asymptotic algebras $\mathcal{A}$ and $\mathcal{A}^s$. Since

$$\mathcal{A} = \mathcal{A}^s \cap \mathcal{A}_{avc}^s \subseteq \mathcal{A}_{avc}^s \subseteq \mathcal{A}_{avc}^w$$

every element A of $\mathcal{A}$ has asymptotically vanishing correlations in the sense that $\langle U_t f \mid A U_t g\rangle \longrightarrow 0$ $(t\to\infty)$ whenever f and g are spatially separating (9.8). We first show that the states on $\mathcal{A}$ and $\mathcal{A}^s$ defined by $U_t f$ and $U_t g$ are pure for all time t and then that these states converge to disjoint states on $\mathcal{A}$ as $t \longrightarrow \infty$.

(10.1) <u>Lemma</u>. For every unit vector f in L^2 let ω_f be the state on $\mathcal{A}$ defined by

$$\omega_f(A) = \langle f \mid Af\rangle \qquad (A \in \mathcal{A})$$

Then each ω_f is pure and ω_f and ω_g are coherent for every pair of unit vectors f and g in L^2. This is also true when $\mathcal{A}$ is replaced by $\mathcal{A}^s$ throughout.

Proof. By (9.20) we have $\tilde{\mathcal{A}}_L \subseteq \mathcal{A} \subseteq \mathcal{A}^S$ and since $\mathcal{A}_L$ is irreducible (8.5), $\mathcal{A}$ and $\mathcal{A}^S$ are also irreducible. The lemma now follows from (A5). ∎

For every f the states $\omega_{U_t f}$ converge as $t \to \infty$ to a state ω_f^∞ on $\mathcal{A}$ (or on $\mathcal{A}^S$) by (9.2). We now investigate when two such nets can converge to disjoint states.

(10.2) Theorem. For each unit vector f in L^2 define a state ω_f^∞ on $\mathcal{A}^s$ by

$$\omega_f^\infty(A) = \lim_{t\to\infty} \langle U_t f \,|\, A U_t f\rangle \qquad (A \in \mathcal{A}^s).$$

Then there are spatially separating vectors f and g in L^2 such that ω_f^∞ and ω_g^∞ are not disjoint.

Proof. By (9.9) there is a unit vector f in L^2 and a Borel set S in $\mathbb{R}^n$ with

$$E_P(S)^\perp U_t^* \mathcal{P} U_t E_P(S) f = \mathcal{P} f$$

and $f = E_P(S) f$ where $\mathcal{P}$ is the parity operator in L^2. Let $g = \mathcal{P} f$ then we have $E_P(S) f = f$ and

$$E_P(\mathbb{R}^n - S) g = E_P(S)^\perp g = g$$

so f and g are spatially separating by (7.3). Now for every $A \in \mathcal{A}^S$

$$\omega_g^\infty(A) = \lim_{t\to\infty} \langle U_t \mathcal{P} f \,|\, A U_t \mathcal{P} f\rangle$$

$$= \omega_f^\infty(\mathcal{P}^{-1} A \mathcal{P})$$

Now by (9.18) we have $\mathcal{P} \in \mathcal{A}^S$ and since $\mathcal{P}$ is unitary it follows that the canonical cyclic representations associated with ω_f^∞ and

ω_g^∞ are unitarily equivalent (see Glimm and Kadison (1960) corollary 8 (note that the hypothesis that the states are pure is not used in the first part of their proof)). Hence ω_f^∞ and ω_g^∞ are not disjoint. ∎

We now investigate some properties of states on the C*-algebra $\mathcal{A}$. Before presenting the main results we prove some lemmas.

(10.3) Lemma. Every element A of $\mathcal{A}$ may be written uniquely in the form $A = X + Y$ for some $X \in \mathcal{A}_0^S$ and some $Y \in L^\infty(P)$. Define a mapping $\pi : \mathcal{A} \to L^\infty(P)$ by $\pi(A) = Y$ whenever $A = X + Y$ with $X \in \mathcal{A}_0^S$ and $Y \in L^\infty(P)$. Then (L^2, π) is a representation of $\mathcal{A}$.

Proof. If $T \in \mathcal{A}_0^S \cap L^\infty(P)$ then T commutes with every U_t so

$$0 = \underset{t \to \infty}{s^*\text{-}\lim}\ U_t^* T U_t = T.$$

Hence $\mathcal{A}_0^S \cap L^\infty(P) = \{0\}$ and the uniqueness of the expression for each A now follows. By (9.18) $\mathcal{A}_0^S$ is an ideal in $\mathcal{A}$ and it is now easily verified that π is a representation. ∎

(10.4) Lemma. Let $f, g \in L^2$. If $\langle f | E_P(R) g \rangle = 0$ for all Borel sets R then there are disjoint Borel sets S_1 and S_2 in $\mathbb{R}^n$ with

$$E_P(S_1)f = f \qquad \text{and} \qquad E_P(S_2)g = g.$$

Proof. Since E_P and E_Q are unitarily equivalent we may prove the result with E_Q replacing E_P. Suppose

$$\langle f | E_Q(R) g \rangle = \int_R \bar{f} g = 0$$

for all R. Let u be the real part of $\bar{f} g$ then $\int_R u = 0$ for all

R. Let u^+ be the positive part of u and define $R^+ = \{x \in \mathbb{R}^n : u(x) > 0\}$ then

$$\int u^+ = \int_{R^+} u^+ = \int_{R^+} u = 0$$

so $u^+ = 0$ (almost everywhere). A similar argument shows $u^- = 0$ (almost everywhere) so $u = 0$ (almost everywhere). Repeating this procedure with u replaced by the imaginary part of $\bar{f}g$ we deduce that $\bar{f}g = 0$ (almost everywhere). Hence $f(x) = 0$ or $g(x) = 0$ for almost all $x \in \mathbb{R}^n$. Let $S_1 = \{x : f(x) \neq 0\}$ and $S_2 = \{x : g(x) \neq 0\}$ then $E_Q(S_1)f = f$, $E_Q(S_2)g = g$ and $S_1 \cap S_2$ has measure zero. The result follows from this. ∎

(10.5) <u>Definition.</u> For each unit vector f in L^2 we denote by ω_f^∞ the state on $\mathcal{A}$ defined by

$$\omega_f^\infty(A) = \lim_{t\to\infty} \langle U_t f \mid A U_t f \rangle \qquad (\forall A \in \mathcal{A}).$$

(This is a state by (9.2).)

(10.6) <u>Theorem.</u> Let f and g be unit vectors in $L^2(\mathbb{R}^n)$ then the states ω_f^∞ and ω_g^∞ on $\mathcal{A}$ are disjoint if and only if there are disjoint Borel sets S_1 and S_2 with

$$E_P(S_1)f = f \qquad \text{and} \qquad E_P(S_2)g = g.$$

<u>Proof.</u> Let π be the representation of $\mathcal{A}$ defined in (10.3) then it follows from the definition of $\mathcal{A}_0^s$ that

$$\omega_f^\infty(A) = \langle f \mid \pi(A) f \rangle$$

and

$$\omega_g^\infty(A) = \langle g \mid \pi(A) g \rangle$$

for all $A \in \mathcal{A}$. Note that $\pi(A) = A$ whenever $A \in L^\infty(P)$ and $\pi(A) = 0$ if $A \in \mathcal{A}_0^s$.

Now if there are no disjoint Borel sets S_1 and S_2 with

$E_p(S_1)f = f$ and $E_p(S_2)g = g$ then by (10.4)

$$\langle f \mid \pi(E_p(R))g \rangle = \langle f \mid E_p(R)g \rangle \neq 0$$

for some R and it follows that ω_f^∞ and ω_g^∞ are not disjoint (4.3).

Conversely suppose $E_p(S_1)f = f$ and $E_p(S_2)g = g$ where S_1 and S_2 are disjoint. Since $\pi(\mathcal{A}) = L^\infty(P)$ it follows that the range of $E_p(S_1)$ is invariant under $\pi(\mathcal{A})$ and we shall denote by π_1 the corresponding subrepresentation of π. Similarly let π_2 be the subrepresentation corresponding to the range of $E_p(S_2)$. Let $\mathcal{B}$ be the von Neumann algebra generated by $\pi(\mathcal{A})$ then $\mathcal{B} = L^\infty(P) = \mathcal{B}'$ and hence $E_p(S_1)$ and $E_p(S_2)$ are equal to their own central supports in $\mathcal{B}'$. Since S_1 and S_2 are disjoint, $E_p(S_1)$ and $E_p(S_2)$ are orthogonal and it follows that π_1 and π_2 are disjoint (Dixmier (1977) p115; see also p375 for further details of central supports).

To complete the proof it is now sufficient to show that the canonical cyclic representations associated with ω_f^∞ and ω_g^∞ are unitarily equivalent to subrepresentations of π_1 and π_2. Let M be the closure of the linear manifold $\{\pi(A)f : A \in \mathcal{A}\} = \{Af : A \in L^\infty(P)\}$. Then $\mathcal{M}$ is invariant under $\pi(\mathcal{A})$ and $\mathcal{M}$ is contained in the range of $E_p(S_1)$. Let π_0 be the corresponding subrepresentation of π_1 then

$$\omega_f^\infty(A) = \langle f \mid \pi_0(A)f \rangle$$

and f is clearly a cyclic vector for π_0. Hence π_0 is unitarily equivalent to the canonical cyclic representation associated with ω_f^∞ (Bratteli and Robinson (1979) p56). A similar argument shows that the canonical cyclic representation associated with ω_g^∞ is unitarily equivalent to a subrepresentation of π_2. ∎

(10.7) Corollary. Let f be a unit vector in L^2 and let S be a Borel set with

$$E_p(S)f \neq 0 \neq E_p(S)^{\perp}f$$

then

$$\omega_f^{\infty} = \| E_p(S)f \|^2 \omega_g^{\infty} + \| E_p(S)^{\perp} f \|^2 \omega_h^{\infty}$$

where $g = \|E_p(S)f\|^{-1} E_p(S)f$ and $h = \|E_p(S)^{\perp}f\|^{-1} E_p(S)^{\perp} f$. Thus each ω_f^{∞} is a mixture of two disjoint states.

Proof. By the theorem ω_g^{∞} and ω_h^{∞} are disjoint and if $A \in \mathcal{A}$ then $A \in \mathcal{A}^S_{avc}$ so

$$\begin{aligned} \omega_f^{\infty}(A) &= \lim_{t\to\infty} \langle f \mid U_t^* A U_t f \rangle \\ &= \lim_{t\to\infty} \langle E_p(S)f \mid U_t^* A U_t E_p(S)f \rangle + \lim_{t\to\infty} \langle E_p(S)^{\perp}f \mid U_t^* A U_t E_p(S)^{\perp}f \rangle \\ &= \| E_p(S)f \|^2 \omega_g^{\infty}(A) + \| E_p(S)^{\perp}f \|^2 \omega_h^{\infty}(A) \end{aligned}$$ ∎

States of the form ω_f^{∞} ($f \in L^2$) may be generalised to give states which are limits of states described by density operators.

(10.8) Theorem. Let W be a density operator on L^2, then the formula

$$\omega(A) = \lim_{t\to\infty} \mathrm{Tr}(W U_t^* A U_t) \qquad (A \in \mathcal{A})$$

defines a state ω on $\mathcal{A}$. If $A = X + Y$ where $X \in \mathcal{A}_0^S$ and $Y \in L^{\infty}(P)$ then

$$\omega(A) = \mathrm{Tr}(WY)$$

In particular ω vanishes on $\mathcal{A}_0^S$.

Proof. Let $A \in \mathcal{A}$ then $A = X+Y$ for some $X \in \mathcal{A}_0^s$ and some $Y \in L^\infty(P)$. Then for every $f \in L^2$ we have

$$\lim_{t\to\infty} \langle f \mid U_t^* A U_t f \rangle = \langle f \mid Yf \rangle$$

Let $\{f_r : r \in \mathbb{N}\}$ be an orthonormal basis for L^2 consisting of eigenvectors of W and for each $r \in \mathbb{N}$ let a_r be the eigenvalue associated with f_r. Since $\sum_{r=1}^\infty a_r = 1$ it follows that $\sum_{r=1}^\infty a_r \langle f_r \mid Yf_r \rangle$ exists and also that

$$\lim_{t\to\infty} \sum_{r=1}^\infty a_r \langle f_r \mid U_t^* X U_t f_r \rangle = \sum_{r=1}^\infty a_r \left(\lim_{t\to\infty} \langle f_r \mid U_t^* X U_t f_r \rangle \right) = 0$$

since the above sum is uniform by the Weierstrass M-test (Apostol (1974) p223). Hence

$$\begin{aligned} \omega\ (A) &= \lim_{t\to\infty} \sum_{r=1}^\infty \langle f_r \mid W U_t^* A U_t f_r \rangle \\ &= \lim_{t\to\infty} \sum_{r=1}^\infty a_r \langle f_r \mid (U_t^* X U_t + Y) f_r \rangle \\ &= \sum_{r=1}^\infty a_r \langle f_r \mid Y f_r \rangle = \mathrm{Tr}\,(WY) \end{aligned}$$

so ω is well defined and $\omega(A) = \mathrm{Tr}(WY)$. Clearly ω is linear and $\omega\ (I) = \sum_{r=1}^\infty a_r = 1$.

Finally since $\mathcal{A}_0^s$ is an ideal in $\mathcal{A}$ (9.18) we have

$$\begin{aligned} \omega\ (A^*A) &= \omega\ (X^*X + X^*Y + Y^*X + Y^*Y) \\ &= \mathrm{Tr}\ (WY^*Y) \\ &\geqslant 0 \end{aligned}$$

so ω is a state on $\mathcal{A}$. ∎

(10.9) Definition. For each density operator W on L^2 we denote by ω_W^∞ the state on $\mathcal{A}$ defined in (10.8). A state ω on $\mathcal{A}$ will be called normal if there is a density operator W on L^2 with $\omega(A) = \mathrm{Tr}(WA)$ for all $A \in \mathcal{A}$.

(10.10) <u>Corollary</u>. If W is a density operator on L^2 then ω_W^∞ is not a normal state on $\mathcal{A}$.

<u>Proof</u>. By way of contradiction suppose there is a density operator W' on L^2 with

$$\omega_W^\infty(A) = Tr(W'A)$$

for all $A \in \mathcal{A}$. Take A to be the projection onto an eigenspace of W' not corresponding to the zero eigenvalue then $Tr(W'A) \neq 0$. But A is finite dimensional and hence compact so $A \in \bar{\mathcal{A}}_L \subseteq \mathcal{A}_0^S$ (by (8.9) and (9.20)) which implies $\omega_W^\infty(A) = 0$ giving the required contradiction. ∎

<u>We shall now assume that the evolution group U in the definition of $\mathcal{A}$ is the free evolution group for a particle of mass m</u>. We then have an immediate corollary to (10.6)

(10.11) <u>Corollary</u>. Let f and g be unit vectors in L^2 then the states ω_f^∞ and ω_g^∞ on $\mathcal{A}$ are disjoint if and only if f and g are spatially separating (with respect to U).

<u>Proof</u>. This follows directly from (7.3) and (10.6). ∎

(10.12) We now discuss the physical implications of reformulating quantum mechanics for a single particle in terms of the C*-algebra $\mathcal{A}$. States of the form ω_W^∞ on $\mathcal{A}$ may be called states at infinity since $\omega_W^\infty(E_Q(S)) = 0$ for all bounded Borel sets S (since $E_Q(S) \in \mathcal{A}_L \subseteq \mathcal{A}_0^S$ (9.20) and ω_W^∞ vanishes on $\mathcal{A}_0^S$ (10.8)). In such a state there is zero probability of a position measurement finding the particle in any bounded region of configuration space so we can imagine the particle to be at "infinity". Such states do not arise in the conventional approach to quantum mechanics where the states are always assumed to be normal.

Although a theory based on $\mathcal{A}$ will contain fewer observables than one based on $\mathbb{B}(L^2)$ there are still sufficient observables to approximate any element of $\mathbb{B}(L^2)$. Since $\mathcal{A}_L \subseteq \mathcal{A}$ (9.20) given $A \in \mathbb{B}(L^2)$ we have $E_Q(S_r)AE_Q(S_r) \to A$ strongly (8.3) where S_r is a ball of radius r centred at the origin in $\mathbb{R}^n$ and $E_Q(S_r)AE_Q(S_r) \in \mathcal{A}$ for each $r \in \mathbb{N}$. Although for example the parity operator $\mathcal{P}$ is no longer an observable we still have local parity operators $E_Q(S)\mathcal{P}E_Q(S)$ in any bounded region S.

Since any state at infinity vanishes on $\mathcal{A}_0^S$ we interpret the observables in $L^\infty(P)$ as the only ones which are relevant to a system at infinity. Hermitian elements of $L^\infty(P)$ will be called (bounded) observables at infinity. Thus the system at infinity is characterised by the commutative von Neumann algebra $L^\infty(P)$. We may picture a state $\omega_{U_t f}$ as breaking up into a mixture of "momentum eigenstates" as $t \to \infty$. (The term "observable at infinity" has been introduced by other authors (Lanford and Ruelle (1969)) with a different meaning.)

(10.13) We now discuss the application of these results to de Broglie's paradox (3.3). Let $h = \frac{1}{\sqrt{2}}(f+g)$ be the state vector of the particle where $\|h\| = \|f\| = \|g\| = 1$ and the states f and g are spatially separating. The state of the particle at time t is then described by the state $\omega_{U_t h}$ on $\mathcal{A}$. Let $A \in \mathcal{A}$ then since f and

g are spatially separating we have by (9.8)

$$\begin{aligned}\omega_h^\infty(A) &= \lim_{t\to\infty} \omega_{U_t h}(A) \\ &= \lim_{t\to\infty} \langle U_t h | A U_t h \rangle \\ &= \tfrac{1}{2} \lim_{t\to\infty} \langle U_t f | A U_t f \rangle + \tfrac{1}{2} \lim_{t\to\infty} \langle U_t g | A U_t g \rangle \\ &= \lim_{t\to\infty} \tfrac{1}{2} (\omega_{U_t f} + \omega_{U_t g})(A) \\ &= \tfrac{1}{2} \omega_f^\infty(A) + \tfrac{1}{2} \omega_g^\infty(A)\end{aligned}$$

Hence for large time t the pure state represented by the state vector $U_t h$ can be approximated by a mixture of the state vectors $U_t f$ and $U_t g$. Now the paradox lies in the assertion in the conventional formulation of quantum mechanics that $U_t h$ remains a pure state at all times and the assertion based on "common sense" that when the two parts $U_t f$ and $U_t g$ are far apart then their correlation should be arbitrarily small so that $U_t f + U_t g$ should be regarded as a mixture as $t \to \infty$. Our present formulation is able to reconcile the above conflicting assertions. The de Broglie paradox no longer exists in our formulation.

Finally we show that our results are not just valid for free evolution groups. By assuming the existence and completeness of the positive wave operator we can define states at infinity for many other evolution groups V and still retain the asymptotic separability of a theory based on the C*-algebra $\mathcal{A}$.

(10.14) Theorem. Let the evolution group U appearing in the definition of $\mathcal{A} = \mathcal{A}_0^S + L^\infty(P)$ (9.17) be an arbitrary free evolution group. Let V be an evolution group in L^2 and assume that the wave operator

$$W_+ = \underset{t\to\infty}{s\text{-}\lim}\ V_t^* U_t$$

exists and is complete. Let f and g be unit vectors in L^2 which are scattering states of V, then we can define states $\bar{\omega}_f^\infty$ and $\bar{\omega}_g^\infty$ on $\mathcal{A}$ by

$$\bar{\omega}_f^\infty(A) = \lim_{t\to\infty} \langle V_t f \mid A V_t f \rangle ,$$

$$\bar{\omega}_g^\infty(A) = \lim_{t\to\infty} \langle V_t g \mid A V_t g \rangle \qquad (A \in \mathcal{A}) .$$

Moreover $\bar{\omega}_f^\infty$ and $\bar{\omega}_g^\infty$ are disjoint if and only if f and g are spatially separating with respect to V.

<u>Proof</u>. First we show that $\bar{\omega}_f^\infty$ and $\bar{\omega}_g^\infty$ are well defined. For a fixed unit vector f in L^2 let ω_f^∞ be the state on $\mathcal{A}$ defined by

$$\omega_f^\infty(A) = \lim_{t\to\infty} \langle U_t f \mid A U_t f \rangle \qquad (A \in \mathcal{A}) .$$

Now let $L = \operatorname*{s-lim}_{t\to\infty} U_t^* A U_t$ then

$$V_t^* A V_t E_\infty(V) = V_t^* U_t (U_t^* A U_t) U_t^* V_t E_\infty(V)$$

$$\xrightarrow{s} W_+ L W_+^* \qquad (t \to \infty) .$$

Hence $\operatorname*{s-lim}_{t\to\infty} V_t^* A V_t E_\infty(V)$ exists and equals $\operatorname*{s-lim}_{t\to\infty} W_+ U_t^* A U_t W_+^*$. Now if f is a scattering state of V then $E_\infty(V) f = f$ and we have

$$\begin{aligned} \bar{\omega}_f^\infty(A) &= \lim_{t\to\infty} \langle V_t f \mid A V_t f \rangle \\ &= \lim_{t\to\infty} \langle f \mid V_t^* A V_t E_\infty(V) f \rangle \\ &= \lim_{t\to\infty} \langle f \mid W_+ U_t^* A U_t W_+^* f \rangle \\ &= \omega_{W_+^* f}^\infty(A) . \end{aligned}$$

It follows that $\bar{\omega}_f^\infty$ is a state on $\mathcal{A}$. Now let f and g be scattering states of V then by the above equality it is sufficient to show that $\omega_{W_+^* f}^\infty$ and $\omega_{W_+^* g}^\infty$ are disjoint if and only if f and g are spatially separating with respect to V. But by (7.2) f and g are spatially separating with respect to V if and only if $W_+^* f$ and $W_+^* g$ are spatially separating with respect to U. The result now follows from (7.3) and (10.6). ∎

To conclude this section we recall from (5.9) that the wave operators exist and are complete for a large class of potentials in L^2. In particular they exist whenever the potential belongs to $L^\infty(Q)$ and hence whenever the potential is a local observable.

11. TWO PARTICLE SYSTEMS AND THE EINSTEIN, PODOLSKY AND ROSEN PARADOX

We now study the separation of states of a system consisting of two distinguishable particles of masses m_1 and m_2. First we fix the notation which will be used throughout this section. The Hilbert space of the combined system will be taken to be $L^2(\mathbb{R}^{2n})$ and the free evolution group U may be written as

$$U_t = U_t^1 \otimes U_t^2$$

where U^i is a free evolution group in $L^2(\mathbb{R}^n)$ for a particle of mass m_i ($i = 1,2$). The position and momentum spectral measures in $L^2(\mathbb{R}^{2n})$ will be denoted by $E_{Q\otimes Q}$ and $E_{P\otimes P}$ and those in $L^2(\mathbb{R}^n)$ by E_Q and E_P. The positions and momenta of the individual particles of the combined system will be described by the spectral measures $E_{Q\otimes I}$, $E_{I\otimes Q}$, $E_{P\otimes I}$, and $E_{I\otimes P}$ as defined in (2.6).

We now introduce the operator algebras which will be used to study the combined system and the individual particles. $L^\infty(P\otimes P)$ will denote the von Neumann algebra on $L^2(\mathbb{R}^{2n})$ consisting of all operators of the form $\int u\, dE_{P\otimes P}$ where $u \in L^\infty(\mathbb{R}^{2n})$ and $L^\infty(P)$ will denote the von Neumann algebra on $L^2(\mathbb{R}^n)$ consisting of all operators of the form $\int u\, dE_P$ where $u \in L^\infty(\mathbb{R}^n)$.

(11.1) Definitions.

$$\mathcal{A}_0^s = \{ A \in \mathbb{B}(L^2(\mathbb{R}^{2n})) : \underset{t\to\infty}{s^*\text{-}\lim}\ U_t^* A U_t = 0 \},$$

$$\mathcal{A} = \mathcal{A}_0^s + L^\infty(P \otimes P),$$

$$\mathcal{A}_0^s(1) = \{ A \in \mathbb{B}(L^2(\mathbb{R}^n)) : \underset{t\to\infty}{s^*\text{-}\lim}\ (U_t^1)^* A U_t^1 = 0 \},$$

$$\mathcal{A}_0^s(2) = \{ A \in \mathbb{B}(L^2(\mathbb{R}^n)) : \underset{t\to\infty}{s^*\text{-}\lim}\ (U_t^2)^* A U_t^2 = 0 \},$$

$$\mathcal{A}(1) = \mathcal{A}_0^s(1) + L^\infty(P),$$

$$\mathcal{A}(2) = \mathcal{A}_0^s(2) + L^\infty(P).$$

Note that since $U_t^1 = U_{m_1 t/m_2}^2$ we have $\mathcal{A}(1) = \mathcal{A}(2)$ so this algebra is in fact independent of the mass of the particle.

In the last section it was shown that de Broglie's paradox could be resolved in an asymptotic sense by assuming that the C*-algebra $\mathcal{A}(1) = \mathcal{A}(2)$ described a single quantum particle. We shall now investigate some properties of the C*-algebra $\mathcal{A}$ defined above and then discuss the physical consequences of assuming that $\mathcal{A}$ describes a system of two distinguishable particles. This will enable us to tackle the Einstein, Podolsky and Rosen paradox in a similar manner to the de Broglie paradox.

(11.2) Theorem.

[1] $\mathcal{A}$ is a proper C*-subalgebra of $\mathbb{B}(L^2(\mathbb{R}^{2n}))$.

[2] If $A \in \mathcal{A}_0^s(1)$ then $A \otimes I$ and $I \otimes A$ both belong to $\mathcal{A}_0^s$.

[3] If $A \in \mathcal{A}(1)$ then $A \otimes I$ and $I \otimes A$ both belong to $\mathcal{A}$.

Proof. [1] follows from (9.18), (9.20) and the definition of $\mathcal{A}$.

[2]: If $A \in \mathcal{A}_0^s(1)$ then for all $f, g \in L^2(\mathbb{R}^n)$

$$\lim_{t\to\infty} \| U_t^*(A\otimes I)U_t(f\otimes g)\| = \lim_{t\to\infty} \|(U_t^1)^* A U_t^1 f\| \|(U_t^2)^* I U_t^2 g\|$$

$$= \|g\| \lim_{t\to\infty} \|(U_t^1)^* A U_t^1 f\|$$

$$= 0 .$$

It now follows that $\lim_{t\to\infty} \|U_t^*(A\otimes I)U_t h\| = 0$ for all h in some dense linear manifold in $L^2(\mathbb{R}^{2n})$. Now let $h \in L^2(\mathbb{R}^{2n})$ be arbitrary, then there is a sequence h_r converging to h with

$$\lim_{t\to\infty} \| U_t^*(A\otimes I)U_t h_r\| = 0$$

for all $r \in \mathbb{N}$. Let u_r and u be defined by

$$u_r(t) = U_t^*(A\otimes I)U_t h_r \qquad \text{and} \qquad u(t) = U_t^*(A\otimes I)U_t h$$

for all $t \in \mathbb{R}$ and all $r \in \mathbb{N}$. Then

$$\sup_{t\in\mathbb{R}} \|u_r(t) - u(t)\| \leq \|A\| \, \|h_r - h\|$$

$$\to 0 \qquad (r\to\infty)$$

so u_r converges uniformly to u and hence (A1) $\lim_{t\to\infty} U_t^*(A\otimes I)U_t h = 0$. Since A is an arbitrary element of $\mathcal{A}(1)$ this limit also equals 0 when A is replaced by A^*. Hence $A\otimes I \in \mathcal{A}_0^s$. Finally since $\mathcal{A}_0^s(1) = \mathcal{A}_0^s(2)$ we have $A \in \mathcal{A}_0^s(2)$ and $I\otimes A \in \mathcal{A}_0^s$ follows from a similar argument.

[3]: Let $A \in \mathcal{A}(1)$ then $A = X+Y$ for some $X \in \mathcal{A}_0^s(1)$ and some $Y \in L^\infty(P)$. Now $Y = \int u \, dE_P$ for some $u \in L^\infty(\mathbb{R}^n)$ and it is easily verified that $Y\otimes I = \int v \, dE_{P\otimes P}$ where $v \in L^\infty(\mathbb{R}^{2n})$ is given by

$$v(x,y) = u(x) \qquad \forall x,y \in \mathbb{R}^n.$$

Hence $Y\otimes I \in L^\infty(P\otimes P)$ Also by [2] we have $X\otimes I \in \mathcal{A}_0^s$ so

$$A\otimes I = X\otimes I + Y\otimes I \in \mathcal{A}_0^s + L^\infty(P\otimes P) = \mathcal{A} .$$

Similarly $I \otimes A$ also belongs to $\mathscr{A}$. ∎

It follows from [2] in the last theorem that $\mathscr{A}$ contains all observables associated with the individual particles. Since $\mathscr{A}$ is a C*-algebra it therefore contains the C*-algebra generated by these one particle observables.

(11.3) Lemma. Let W be a density operator on $L^2(\mathbb{R}^{2n})$. Then we can define a state ω on $\mathscr{A}$ by

$$\omega(A) = \lim_{t \to \infty} \mathrm{Tr}(W U_t^* A U_t) \qquad (\forall A \in \mathscr{A}).$$

If $A = X + Y$ where $X \in \mathscr{A}_0^S$ and $Y \in L^\infty(P \otimes P)$ then

$$\omega(A) = \mathrm{Tr}(WY),$$

so in particular ω vanishes on $\mathscr{A}_0^S$. If S is a bounded Borel set in $\mathbb{R}^n$ then

$$\omega(E_{Q \otimes I}(S)) = 0 = \omega(E_{I \otimes Q}(S)).$$

Proof. The first part of the lemma follows from (10.8). If is bounded then $E_Q(S) \in \mathscr{A}_L$ where $\mathscr{A}_L$ is the *-subalgebra of $\mathcal{B}(L^2(\mathbb{R}^n))$ defined in (8.1). By (9.20) we have $E_Q(S) \in \mathscr{A}_0^S(1)$ and now $E_{Q \otimes I}(S) = E_Q(S) \otimes I \in \mathscr{A}_0^S$ by (11.2). Hence

$\omega(E_{Q \otimes I}(S)) = 0$ and similarly $\omega(E_{I \otimes Q}(S)) = 0$. ∎

(11.4) Definition. If W is a density operator on $L^2(\mathbb{R}^{2n})$ then we denote by ω_W^∞ the state on $\mathscr{A}$ defined in (11.3). If h is a unit vector in $L^2(\mathbb{R}^{2n})$ then we shall write ω_h^∞ for the state ω_W^∞ obtained by taking W to be the projection onto the subspace spanned by h.

It follows from (11.3) that in any state of the form ω_w^∞ the probability of a position measurement finding either particle in some bounded region in $\mathbb{R}^n$ is zero. Hence we shall call each ω_w^∞ a state at infinity. In such a state both of the particles are at "infinity". Since any state at infinity vanishes on $\mathcal{A}_0^S$ we shall regard the observables in $L^\infty(P\otimes P)$ as being the only relevant observables of the system when it is at infinity. Hermitian elements of $L^\infty(P\otimes P)$ will be called observables at infinity. Disjointness of states at infinity can be characterised by (10.6) which we now rewrite in the notation of this section.

(11.5) Theorem. Let f and g be unit vectors in $L^2(\mathbb{R}^{2n})$ then the states ω_f^∞ and ω_g^∞ on $\mathcal{A}$ are disjoint if and only if there are disjoint Borel sets R and S in $\mathbb{R}^{2n}$ with

$$E_{P\otimes P}(R)f = f \qquad \text{and} \qquad E_{P\otimes P}(S)g = g .$$

Proof. This follows immediately from (10.6) and the fact that n is arbitrary throughout section 10. ∎

When the vectors f and g in (11.5) are "tensor products" of elements of $L^2(\mathbb{R}^n)$ we have the following relation between disjointness and spatial separation.

(11.6) Theorem. Let f_1, f_2, g_1 and g_2 be unit vectors in $L^2(\mathbb{R}^n)$ then the states $\omega_{f_1\otimes f_2}^\infty$ and $\omega_{g_1\otimes g_2}^\infty$ are disjoint if and only if f_1 and g_1 are spatially separating with respect to U^1 or f_2 and g_2 are spatially separating with respect to U^2.

Proof. First suppose $\omega^{\infty}_{f_1 \otimes f_2}$ and $\omega^{\infty}_{g_1 \otimes g_2}$ are disjoint. Then by (11.5) there are disjoint Borel sets R and S in $\mathbb{R}^{2n}$ with

$$E_{P\otimes P}(R)(f_1 \otimes f_2) = f_1 \otimes f_2 \qquad \text{and} \qquad E_{P\otimes P}(S)(g_1 \otimes g_2) = g_1 \otimes g_2 .$$

Let B_1 and B_2 be Borel sets in $\mathbb{R}^n$ then

$$\begin{aligned} & \langle f_1 \otimes f_2 \mid E_{P\otimes P}(B_1 \times B_2)(g_1 \otimes g_2)\rangle \\ = \quad & \langle E_{P\otimes P}(R)(f_1 \otimes f_2) \mid E_{P\otimes P}(B_1 \times B_2) E_{P\otimes P}(S)(g_1 \otimes g_2)\rangle \\ = \quad & \langle f_1 \otimes f_2 \mid E_{P\otimes P}(R \cap (B_1 \times B_2) \cap S)(g_1 \otimes g_2) \\ = \quad & 0 . \end{aligned}$$

Hence for all Borel sets B_1 and B_2 in $\mathbb{R}^n$ we have

$$\langle f_1 \mid E_P(B_1) g_1 \rangle \langle f_2 \mid E_P(B_2) g_2 \rangle = 0 .$$

Assume that f_1 and g_1 are not spatially separating then by (7.3) and (10.4) we have $\langle f_1 \mid E_P(B_1) g_1 \rangle \neq 0$ for some Borel set B_1 in $\mathbb{R}^n$ and it follows that $\langle f_2 \mid E_P(B_2) g_2 \rangle = 0$ for all Borel sets B_2 in $\mathbb{R}^n$. Now applying (10.4) and (7.3) again we deduce that f_2 and g_2 are spatially separating.

This completes the proof of the first part of the theorem. To prove the converse implication suppose f_1 and g_1 are spatially separating. Then by (7.3) there are disjoint Borel sets B_1 and B_2 with

$$E_P(B_1) f_1 = f_1 \qquad \text{and} \qquad E_P(B_2) g_1 = g_1 .$$

Let $R = B_1 \times \mathbb{R}^n$ and $S = B_2 \times \mathbb{R}^n$ then R and S are disjoint and

$$\begin{aligned} E_{P\otimes P}(R)(f_1 \otimes f_2) &= (E_P(B_1) f_1) \otimes f_2 = f_1 \otimes f_2 , \\ E_{P\otimes P}(S)(g_1 \otimes g_2) &= (E_P(B_2) g_1) \otimes g_2 = g_1 \otimes g_2 . \end{aligned}$$

Hence $\omega^{\infty}_{f_1 \otimes f_2}$ and $\omega^{\infty}_{g_1 \otimes g_2}$ are disjoint by (11.5) and a similar argument shows that these two states are disjoint if f_2 and g_2 are spatially separating. ∎

(11.7) <u>Theorem</u>. Let $h = a(f_1 \otimes f_2) + b(g_1 \otimes g_2)$ where f_1, f_2, g_1 and g_2 are unit vectors in $L^2(\mathbb{R}^n)$ and $a, b \in \mathbb{C}$ with $|a|^2 + |b|^2 = 1$. Suppose f_1 and g_1 are eigenvectors of some element of $L^\infty(P)$ corresponding to distinct eigenvalues then the states $\omega^\infty_{f_1 \otimes f_2}$ and $\omega^\infty_{g_1 \otimes g_2}$ are disjoint and

$$\omega^\infty_h = |a|^2 \omega_{f_1 \otimes f_2} + |b|^2 \omega_{g_1 \otimes g_2} .$$

<u>Proof</u>. Since f_1 and g_1 are eigenvectors of some element of $L^\infty(P)$ corresponding to distinct eigenvalues there are disjoint Borel sets R and S with $E_P(R)f_1 = f_1$ and $E_P(S)g_1 = g_1$. Now $R \times \mathbb{R}^n$ and $S \times \mathbb{R}^n$ are disjoint and

$$E_{P \otimes P}(R \times \mathbb{R}^n)(f_1 \otimes f_2) = f_1 \otimes f_2$$

and

$$E_{P \otimes P}(S \times \mathbb{R}^n)(g_1 \otimes g_2) = g_1 \otimes g_2 .$$

It now follows from (11.5) that $\omega^\infty_{f_1 \otimes f_2}$ and $\omega^\infty_{g_1 \otimes g_2}$ are disjoint and the last relation in the statement of the theorem is easily verified. ∎

(11.8) We now discuss the Einstein, Podolsky and Rosen paradox (3.2) from the viewpoint of an algebraic theory in which $\mathcal{A}$ is taken as the C*-algebra of the two particle system. Suppose that at time $t = 0$ the state vector of the two particle system is

$$h = a(f_1 \otimes f_2) + b(g_1 \otimes g_2) .$$

Then the state vector at time t is given by

$$U_t h = a U_t(f_1 \otimes f_2) + b U_t(g_1 \otimes g_2) .$$

Conventionally in arriving at the Einstein, Podolsky and Rosen paradox we imagine a measurement of some observable $A \otimes I$ associated with the first system at a time t when the two particles are far apart. The vectors f_1 and g_1 are chosen so that $U_t^1 f$ and $U_t^1 g$ are eigenvectors of A corresponding to distinct eigenvalues.

We shall assume that the measurement of A takes place after an infinite time (in practice a large time) has elapsed (e.g. since we defined spatially separating systems in an asymptotic sense (7.6) it may take an infinite time for the two particles to separate). Now after an infinite time has elapsed both particles are at infinity so in particular the only relevant observables of the first particle are the observables at infinity in $L^{\infty}(P)$ and hence we must have $A \in L^{\infty}(P)$. It is only possible for a measurement of $A \otimes I$ to distinguish the two states $f_1 \otimes f_2$ and $g_1 \otimes g_2$ (and thereby lead to a paradox in the usual formulation of quantum mechanics) if f_1 and g_1 correspond to distinct eigenvalues.

However in our approach if f_1 and g_1 correspond to distinct eigenvalues of $A \in L^{\infty}(P)$ then by (11.7) as $t \to \infty$ the state described by the wave function $U_t h$ converges to ω_h^{∞} which is a mixture of the disjoint states $\omega^{\infty}_{f_1 \otimes f_2}$ and $\omega^{\infty}_{g_1 \otimes g_2}$. Thus in our algebraic formulation based on the C*-algebra $\mathcal{A}$ the Einstein, Podolsky and Rosen paradox may be tackled in a similar asymptotic fashion to the de Broglie paradox.

Finally the results of this section may be extended to more general evolution groups provided we assume the existence and completeness of a suitable wave operator

(11.9) <u>Theorem</u> Let V be an evolution group in $L^2(\mathbb{R}^{2n})$ such that the wave operator

$$W_+ = \underset{t \to \infty}{\text{s-lim}}\; V_t^* U_t$$

exists and is complete. Then for every scattering state f of V we can define a state $\overline{\omega}_f^{\infty}$ on $\mathcal{A}$ by

$$\overline{\omega}_f^{\infty}(A) = \lim_{t\to\infty} \langle V_t f \mid A V_t f \rangle \qquad (\forall A \in \mathcal{A})$$

and for such an f we have

$$\overline{\omega}_f^{\infty} = \omega^{\infty}_{W_+^* f} .$$

Proof This follows from the first part of the proof of (10.14). ∎

This result enables us to give a similar resolution to the Einstein, Podolsky and Rosen Paradox when the two particles are no longer free. A superposition of two coherent state vectors of the form considered in the derivation of the paradox will converge to a mixture of two disjoint states as $t \to \infty$.

CHAPTER V. A FURTHER STUDY OF LOCAL OBSERVABLES.

12. A GENERALISED FORM OF LOCAL OBSERVABLES.

In this chapter we shall study a generalisation of the local observables of Wan and Jackson (1983) which were discussed in section 8. We shall define observables which are localised with respect to a general spectral measure E defined on the Borel set of $\mathbb{R}^k$ for some $k \in \mathbb{N}$ and whose values are projections in a Hilbert space $\mathcal{H}$. In particular E may be the position spectral measure in $L^2(\mathbb{R}^k)$ or the position spectral measure of a particle with spin. Alternatively if E is a momentum spectral measure then we are establishing a "localisation" in the momentum space of the system. If $k = 1$ then we have a "localisation" in the spectrum of the observable corresponding to E. We shall concentrate our study on some mathematical properties of these generalised local observables; a study of the physical aspects is being undertaken by Wan and Jackson.

Let $A \in \mathbb{B}(\mathcal{H})$ be Hermitian then we shall say that A is localised with respect to E(R) if the following conditions are both satisfied,

[1] if the state vector of the system lies in the range of E(R) before a measurement of A then it lies in the range of E(R) after the measurement,

[2] if the state vector of the system lies in the range of $E(R)^{\perp}$ before a measurement of A then the state is not affected by a measurement of A.

In the case where E represents the position of a quantum mechanical system then the interpretation of [1] is that if the system is located inside R before a measurement of A then it remains localised in R after the measurement. Also [2] is interpreted as meaning that if the system is outside R before the measurement then the state is unaffected by the measurement. Physically we can then interpret R as being related to the size of the measuring instrument used to measure A.

To obtain a precise formulation of [1] and [2] we shall assume that the change of state during a measurement is described by Lüdders' Rule (1.6). Then if f is the state vector of the system before a measurement of an observable A then the state vector f' after the measurement is given by

$$f' = \| E_A(S)f \|^{-1} E_A(S)f$$

where E_A is the spectral measure of A and S is the Borel set in $\mathbb{R}$ in which the measured value of A lies.

Combining this formula for the change of state and conditions [1] and [2] above we may give the following definition of an observable which is localised with respect to $E(R)$:

(12.1) <u>Definition</u>. Let A be a Hermitian operator on $\mathcal{H}$ and let E_A be the spectral measure of A. If R is a Borel set in $\mathbb{R}^k$ then we shall say that A is localised with respect to $E(R)$ if the following conditions hold for all unit vectors f in $\mathcal{H}$

[1] if $E(R)f = f$ then for all Borel sets S in $\mathbb{R}$

$E(R)E_A(S)f = E_A(S)f$,

[2] if $E(R)f = 0$ then for all Borel sets S in $\mathbb{R}$

$E_A(S)f = \|E_A(S)f\| f$.

(12.2) Theorem. Let A be a Hermitian operator on $\mathcal{H}$ with spectral measure E_A. If $A = E(R)XE(R) + aI$ for some $X \in \mathcal{B}(\mathcal{H})$ some $a \in \mathbb{C}$ and some Borel set R in $\mathbb{R}^n$ then A is localised with respect to $E(R)$.

Proof. [1]: Since $E(R)$ clearly commutes with A it also commutes with $E_A(S)$ for every Borel set S in $\mathbb{R}$ (see (A4)). Now if $f \in \mathcal{H}$ and $E(R)f = f$ then for every Borel set S in $\mathbb{R}$ we have

$$\begin{aligned} E(R)E_A(S)f &= E_A(S)E(R)f \\ &= E_A(S)f \end{aligned}$$

so property [1] of (12.1) holds.

[2]: Let $E_A\{a\}$ be an abuse of notation for $E_A(\{a\})$ then $E_A\{a\}$ is the projection onto the subspace $\{f \in \mathcal{H} : Af = af\}$ (Dunford and Schwartz (1963) p904). If $E(R)f = 0$ then

$$Af = E(R)XE(R)f + af = af$$

and it follows that $E(R)^{\perp} \leq E_A\{a\}$. To prove property [2] of (12.1) let S be a Borel set in $\mathbb{R}$. We consider the cases $a \notin S$ and $a \in S$ separately.

(i) Suppose $a \notin S$ and let $f \in \mathcal{H}$ with $E(R)f = 0$. Since $a \notin S$ the projections $E_A\{a\}$ and $E_A(S)$ are orthogonal so

$$E(R)^{\perp} \leq E_A\{a\} \leq E_A(S)^{\perp}$$

giving $E_A(S) \leq E(R)$. Hence

$$\|E_A(S)f\|^2 \leq \|E(R)f\|^2 = 0$$

so $E_A(S)f = 0$. Now we clearly have $E_A(S)f = 0 = \|E_A(S)f\| f$ so [2] holds in this case.

(ii) Suppose $a \in S$ and let $f \in \mathcal{H}$ with $\|f\| = 1$ and $E(R)f = 0$. Let $S' = S - \{a\}$ then $E_A(S) = E_A(S') + E_A\{a\}$ and since $a \notin S'$ we have $E_A(S')f = 0$ as in (i) above. Now

$$E_A(S)f = E_A(S')f + E_A\{a\}f = E_A\{a\}f.$$

But $E_A\{a\}^{\perp} \leq E(R)$ and $E(R)f = 0$ so $E_A\{a\}^{\perp}f = 0$ giving $E_A\{a\}f = f$ and hence

$$E_A(S)f = E_A\{a\}f = f.$$

Since $\|f\| = 1$ this implies $E_A(S)f = \|E_A(S)f\| f$ in this case also. ∎

(12.3) Lemma. Let M be a projection in a Hilbert space $\mathcal{H}$ and let $A \in \mathbb{B}(\mathcal{H})$ and $a \in \mathbb{C}$ Then these are equivalent

[1] $A = MXM + aI$ for some $X \in \mathbb{B}(\mathcal{H})$;

[2] $A = MXM + aM^{\perp}$ for some $X \in \mathbb{B}(\mathcal{H})$;

[3] $A = MAM + aM^{\perp}$.

If A is Hermitian and $a \in \mathbb{R}$ then each of these is also equivalent to

[4] $Af = af$ for all f in the range of $M^{\perp}$.

Proof. ([1] $\Rightarrow$ [2]) This follows from the identity

$$MXM + aI = M(X + aM)M + aM^{\perp}$$

([2] $\Rightarrow$ [3]) If $A = MXM + aM^{\perp}$ then

$$\begin{aligned} MAM + aM^{\perp} &= M(MXM + aM^{\perp})M + aM^{\perp} \\ &= MXM + aM^{\perp} \\ &= A. \end{aligned}$$

Since [3] $\Rightarrow$ [1] is obvious the first three statements are equivalent and we now assume A is Hermitian and $a \in \mathbb{R}$.

($[3] \Rightarrow [4]$) If $A = MAM + aM^{\perp}$ and $f \in \text{ran}(M^{\perp})$ then $Mf = 0$ and $Af = aM^{\perp}f = af$.

($[4] \Rightarrow [3]$) It follows from [4] that $A(M^{\perp}f) = a(M^{\perp}f)$ for all f in $\mathcal{H}$ so $AM^{\perp} = aM^{\perp}$ and (taking adjoints) $M^{\perp}A = aM^{\perp}$. Now

$$\begin{aligned} A &= (M + M^{\perp})A(M + M^{\perp}) \\ &= MAM + M^{\perp}AM + MAM^{\perp} + M^{\perp}AM^{\perp} \\ &= MAM + aM^{\perp}M + aMM^{\perp} + aM^{\perp} \\ &= MAM + aM^{\perp}. \end{aligned}$$ ∎

It will now be shown that every observable which is localised with respect to $E(R)$ is of the form given in the last theorem. Note that the lemma above gives a number of different ways of expressing these observables.

(12.4) Theorem. Let A be a Hermitian operator on $\mathcal{H}$. Then A is localised with respect to $E(R)$ if and only if $A = E(R)AE(R) + aE(R)^{\perp}$ for some $a \in \mathbb{R}$.

Proof. If A is of the given form then it follows from (12.2) and (12.3) that A is localised with respect to $E(R)$.

Conversely suppose that A is localised with respect to $E(R)$. Let f be a unit vector in $\mathcal{H}$, then we have from the definition (12.1) that for every Borel set S in $\mathbb{R}$

$$[1] \; E(R)f = f \;\Rightarrow\; E(R)E_A(S)f = E_A(S)f$$

and $$[2] \; E(R)f = 0 \;\Rightarrow\; E_A(S)f = \|E_A(S)f\| f\,.$$

From property [1] it follows that if f is a unit vector in the range of $E(R)$ then $E_A(S)f$ belongs to the range of $E(R)$ so $E_A(S)E(R) = E(R)E_A(S)E(R)$. Taking adjoints of each side of this equality we see that $E(R)$ commutes with $E_A(S)$ and since S is arbitrary $E(R)$ commutes with A (see (A4)). Hence

$$A = (E(R) + E(R)^{\perp})A(E(R) + E(R)^{\perp})$$
$$= E(R)AE(R) + AE(R)^{\perp}.$$

To complete the proof [2] will be used to show that $AE(R)^{\perp} = aE(R)^{\perp}$ for some $a \in \mathbb{R}$. We recall that $Af = af$ if and only if $E_A\{a\}f = f$ (see the proof of (12.2)).

Let f be a unit vector in the range of $E(R)^{\perp}$ then by [2] $E_A(S)f = \|E_A(S)f\|E_A(S)f$ so $\|E_A(S)f\| \in \{0,1\}$ for all Borel sets S. Fixing such an f and defining $h: \mathbb{R} \to \mathbb{R}$ by

$$h(t) = \|E_A((-\infty,t])f\|^2$$

it follows that h is increasing, right continuous, and takes the values 0 and 1 for suitably large negative and positive t (Prugovecki (1981) pp235-236). Hence h has a unique discontinuity at some point $a \in \mathbb{R}$ and

$$\|E_A\{a\}f\|^2 = \|E_A((-\infty,a])f\|^2 - \|E_A((-\infty,a))f\|^2$$
$$= h(a) - \lim_{n\to\infty} h(a - 1/n)$$
$$= 1$$

so $E_A\{a\}f = f$ and hence $Af = af$. Now let f and g be non-zero vectors in the range of $E(R)$, then there are constants a,b,c with $Af = af$, $Ag = bg$, and $A(f + g) = c(f + g)$. If f and g are linearly dependent clearly $a = b$ and otherwise

$$af + bg = A(f + g) = cf + cg$$

so $a = c = b$. It follows that $AE(R)^{\perp} = aE(R)^{\perp}$ as required. ∎

(12.5) <u>Theorem</u>. Let M be a projection on $\mathcal{H}$ and let $\mathbb{B}$ denote the von Neumann algebra $\mathbb{B}(\mathcal{H})$ Then $M\mathbb{B}M$ is a C*-subalgebra of $\mathbb{B}$ and the von Neumann algebra generated by $M\mathbb{B}M$ is

$$(M\mathbb{B}M)'' = M\mathbb{B}M + \mathbb{C}M^{\perp} = M\mathbb{B}M + \mathbb{C}I.$$

A Hermitian $A \in \mathbb{B}$ belongs to $(M\mathbb{B}M)''$ if and only if $A = MAM + aM^{\perp}$ for some $a \in \mathbb{R}$.

<u>Proof</u>. The proof that $M\mathbb{B}M$ is a C*-subalgebra is identical to the proof of (8.2).

To prove $(M\mathbb{B}M)'' = M\mathbb{B}M + \mathbb{C}M^{\perp}$ we begin by showing

$$(M\mathbb{B}M)' = M^{\perp}\mathbb{B}M^{\perp} + \mathbb{C}M \qquad (*)$$

If $X = M^{\perp}XM^{\perp} + aM$ and $Y = MYM$ then $XY = aM = YX$ and this implies $M^{\perp}\mathbb{B}M^{\perp} + \mathbb{C}M \subseteq (M\mathbb{B}M)'$. Conversely suppose $X \in (M\mathbb{B}M)'$ then X commutes with M and so

$$X = (M + M^{\perp})X(M + M^{\perp}) = MXM + M^{\perp}XM^{\perp}.$$

Also MXM commutes with MAM for every $A \in \mathbb{B}$ so the restriction of MXM to ran(M) commutes with every bounded operator on ran(M) and it follows that $MXM = aM$ for some $a \in \mathbb{C}$. Hence $X = M^{\perp}XM + aM$ so $(M\mathbb{B}M)' \subseteq M^{\perp}\mathbb{B}M^{\perp} + \mathbb{C}M$ and (*) has been verified. Now two applications of (*) give

$$\begin{aligned} A \in (M\mathbb{B}M)'' &\Leftrightarrow A \in (M^{\perp}\mathbb{B}M^{\perp} + \mathbb{C}M)' \\ &\Leftrightarrow A(M^{\perp}XM^{\perp} + aM) = (M^{\perp}XM^{\perp} + aM)A \quad (\forall X \in \mathbb{B})(\forall a \in \mathbb{C}) \\ &\Leftrightarrow AM^{\perp}XM^{\perp} = M^{\perp}XM^{\perp}A \qquad (\forall X \in \mathbb{B}) \\ &\Leftrightarrow A \in (M^{\perp}\mathbb{B}M^{\perp})' \\ &\Leftrightarrow A \in M\mathbb{B}M + \mathbb{C}M^{\perp}. \end{aligned}$$

Finally the last assertion in the statement of the theorem follows from (12.3). ∎

Taking $M = E(R)$ we see that the observables which are localised with respect to $E(R)$ can be represented by the Hermitian elements of the von Neumann algebra $(E(R)\mathbb{B}E(R))'' = E(R)\mathbb{B}E(R) + \mathbb{C}I$.

When E is the position spectral measure in $L^2(\mathbb{R}^k)$ this is just the von Neumann algebra generated by the local observables in R (in the sense of Wan and Jackson (1983)). In the algebraic approach to quantum field theory it is often regarded as convenient from a mathematical point of view to replace the C*-algebra of observables which can be measured in a region R by the von Neumann algebra they generate (see Haag and Kastler (1964) p848). Thus when E is the position spectral measure our local observables (although more general) can be obtained in a natural way from those of Wan and Jackson.

13. THE ALGEBRA OF ALL LOCAL OBSERVABLES IN THE GENERALISED SENSE.

Throughout this section E will denote a spectral measure on the Borel sets of $\mathbb{R}^k$ whose values are projections on the Hilbert space $\mathcal{H}$. From the last section Hermitian operators of the form $A = E(R)AE(R) + aE(R)^{\perp}$ $(a \in \mathbb{R})$ represent observables which are localised with respect to $E(R)$. We shall say that an element of $\mathbb{B}(\mathcal{H})$ is E-local if it is of the above form where R is a bounded Borel set. In this section we investigate the *-algebra generated by all E-local observables and show that in general this *-algebra is not closed and is not dense in $\mathbb{B}(\mathcal{H})$ in the operator norm.

(13.1) <u>Definition.</u> Define $\mathcal{A}_L(E)$ to be the set of all operators of the form $E(R)AE(R)$ where R is a bounded Borel set in $\mathbb{R}^k$ and $A \in \mathbb{B}(\mathcal{H})$. $\mathcal{A}_I(E)$ will denote the set of all operators of the form $E(R)AE(R) + aE(R)^{\perp}$ where R is a bounded Borel set, $A \in \mathbb{B}(\mathcal{H})$ and $a \in \mathbb{C}$. The closures of $\mathcal{A}_L(E)$ and $\mathcal{A}_I(E)$ in the operator norm will be denoted by $\bar{\mathcal{A}}_L(E)$ and $\bar{\mathcal{A}}_I(E)$ respectively.

When there is no danger of confusion over the spectral measure involved we shall just write $\mathcal{A}_L$, $\mathcal{A}_I$, $\bar{\mathcal{A}}_L$ and $\bar{\mathcal{A}}_I$ to denote these sets.

(13.2) Lemma.

[1] Let $A \in \mathbb{B}(\mathcal{H})$ then $A \in \mathcal{A}_L$ if and only if $A = E(R)AE(R)$ for some bounded Borel set R.

[2] Let $A \in \mathbb{B}(\mathcal{H})$ then these are equivalent

[a] $A \in \mathcal{A}_I$;

[b] $A = E(R)AE(R) + aE(R)^{\perp}$ for some bounded Borel set R and some $a \in \mathbb{C}$;

[c] $A = E(R)XE(R) + aI$ for some bounded Borel set R, some $X \in \mathbb{B}(\mathcal{H})$ and some $a \in \mathbb{C}$;

[d] $A - aI \in \mathcal{A}_L$ for some $a \in \mathbb{C}$

Proof. Those assertions which do not follow from (12.3) are obvious. ∎

(13.3) Theorem. $\mathcal{A}_L$ and $\mathcal{A}_I$ are irreducible *-subalgebras of $\mathbb{B}(\mathcal{H})$ and the von Neumann algebra generated by each of these *-subalgebras is equal to $\mathbb{B}(\mathcal{H})$. Also

$$\mathcal{A}_I = \mathcal{A}_L + \mathbb{C}I$$

and

$$\overline{\mathcal{A}_I} = \overline{\mathcal{A}_L} + \mathbb{C}I.$$

Proof. The proof that $\mathcal{A}_L$ is an irreducible *-subalgebra of $\mathbb{B}(\mathcal{H})$ and that $\mathbb{B}(\mathcal{H})$ is the von Neumann algebra generated by $\mathcal{A}_L$ is obtained by replacing E_Q by E in the relevant parts of the proofs of (8.3) and (8.5). The corresponding assertions for $\mathcal{A}_I$ follow from the identity $\mathcal{A}_I = \mathcal{A}_L + \mathbb{C}I$ which is a consequence of [2] in the lemma. The identity $\overline{\mathcal{A}_I} = \overline{\mathcal{A}_L} + \mathbb{C}I$ follows from $\mathcal{A}_I = \mathcal{A}_L + \mathbb{C}I$ and the fact that $\overline{\mathcal{A}_L} + \mathbb{C}I$ is closed (e.g. Berberian (1974) p90). ∎

The properties of the *-algebras $\mathcal{A}_L$ and $\mathcal{A}_I$ are closely related to the spectrum of the spectral measure E (see (A6) for the definition and basic properties of the spectrum of a spectral measure). We shall call a spectral measure compact if its spectrum is compact.

(13.4) Lemma. Suppose E is not a compact spectral measure on the Borel sets of $\mathbb{R}^k$. Then the spectrum of E is the union of a disjoint sequence S_r of bounded Borel sets such that for each r

[1] $E(S_r) \neq 0$,

[2] S_r lies outside a closed ball of radius $r-1$ centred at the origin in $\mathbb{R}^k$.

Proof. Let Λ be the spectrum of E then $E(\Lambda) = I$ (A6) and by hypothesis Λ is not compact. Define

$$A_j = \{ x \in \mathbb{R}^k : j-1 \leq |x| < j \} \qquad (\forall j \in \mathbb{N}).$$

Then for any subset S of $\mathbb{R}^k$, $S \cap A_j$ is a disjoint sequence of bounded Borel sets with union S.

Let $R_j = \Lambda \cap A_j$. If $E(R_j) \neq 0$ for only finitely many j then $E(\Lambda) = E(R)$ for some bounded R and hence

$$I = E(\Lambda) = E(R) \geq E(\overline{R})$$

where $\overline{R}$ is the closure of R. Now $\Lambda \subseteq \overline{R}$ so Λ is bounded which is a contradiction. Hence we may form the subsequence R_{j_r} $(r \in \mathbb{N})$ consisting of those R_j for which $E(R_j) \neq 0$. Let D be the union of all those R_j with $E(R_j) = 0$ and define

$$S_r = R_{j_r} \cup (D \cap A_r).$$

Then S_r is a sequence with the required properties. ∎

(13.5) Theorem. These are equivalent :

[1] E is a compact spectral measure;

[2] $\mathscr{A}_L = \mathscr{A}_I$;

[3] $\mathscr{A}_L = \mathbb{B}(\mathcal{H})$;

[4] $\mathscr{A}_I = \mathbb{B}(\mathcal{H})$;

[5] $\mathscr{A}_L$ is closed in $\mathbb{B}(\mathcal{H})$;

[6] $\mathscr{A}_I$ is closed in $\mathbb{B}(\mathcal{H})$.

Proof. It is sufficient to prove the implications in the following diagram:

$$\begin{tikzcd}
 & {[2]} \arrow[dr, Rightarrow] & \\
{[1]} \arrow[ur, Rightarrow] & & {[3]} \arrow[dl, Rightarrow] \arrow[dd, Rightarrow] \\
 & {[5]} \arrow[dl, Rightarrow] & \\
{[6]} \arrow[uu, Rightarrow] & & {[4]} \arrow[ll, Rightarrow]
\end{tikzcd}$$

All those lying on or below the diagonal are either obvious or follow immediately from $\mathscr{A}_I = \mathscr{A}_L + \mathbb{C}I$ and $\bar{\mathscr{A}}_I = \bar{\mathscr{A}}_L + \mathbb{C}I$ (13.3) so we prove the remaining three.

([1] $\Rightarrow$ [2]) Suppose E is compact then the spectrum Λ of E is a bounded Borel set with $E(\Lambda) = I$. Hence $I \in \mathscr{A}_L$ so $\mathscr{A}_L + \mathbb{C}I = \mathscr{A}_L$ and [2] follows.

([2] $\Rightarrow$ [3]) If $\mathscr{A}_L = \mathscr{A}_I$ then $I \in \mathscr{A}_L$ which implies $I = E(R)$ for some bounded R and now

$$\mathbb{B}(\mathcal{H}) = E(R)\,\mathbb{B}(\mathcal{H})\,E(R) \subseteq \mathscr{A}_L$$

([6] $\Rightarrow$ [1]) We assume E is not compact and show that $\mathscr{A}_I$ is not closed. Let S_r be a disjoint sequence of bounded Borel set satisfying properties [1] and [2] of the lemma. Now the partial sums of the series $\sum 2^{-r} E(S_{2r})$ belong to $\mathscr{A}_L$ and since this series is clearly absolutely convergent in $\mathbb{B}(\mathcal{H})$ it converges to an element A of $\bar{\mathcal{A}}_L$. Clearly $A \in \bar{\mathcal{A}}_L + \mathbb{C}I = \bar{\mathcal{A}}_I$ and we show by contradiction that $A \notin \mathcal{A}_I$.

Suppose A does belong to $\mathcal{A}_I$ then $A = E(R)AE(R) + aE(R)^{\perp}$ for some bounded Borel set R and some $a \in \mathbb{C}$ so

$$\sum_{r=1}^{\infty} 2^{-r} E(S_{2r}) = \sum_{r=1}^{\infty} 2^{-r} E(R \cap S_{2r}) + aE(R)^{\perp}.$$

It follows from property 2 of the lemma that $R \cap S_{2j+1} = \phi$ for some j so multiplying each side of the above equality by $E(S_{2j+1})$ and using the fact that S_r is a disjoint sequence we have

$$0 = aE(S_{2j+1})$$

and it follows from property [1] of the lemma that $a = 0$ Hence for all f in $\mathcal{H}$

$$\sum_{r=1}^{\infty} 2^{-r} E(S_{2r}) f = \sum_{r=1}^{\infty} 2^{-r} E(R \cap S_{2r}) f.$$

Now by property [2] of the lemma $R \cap S_{2i} = \phi$ for some $i \in \mathbb{N}$ and by [1]of the lemma there is a non-zero vector f in the range of $E(S_{2i})$. For such an f the last identity gives

$$2^{-i} f = 0$$

which is a contradiction since $f \neq 0$. We conclude that $A \in \bar{\mathcal{A}}_I - \mathcal{A}_I$ so $\mathcal{A}_I$ is not closed. ∎

14. GENERALISED LOCAL OBSERVABLES AND QUANTUM LOGIC.

In this section we shall study some properties of projections in the *-algebras $\mathcal{A}_I$ and $\bar{\mathcal{A}}_I$ defined in (13.1). We shall assume that these *-algebras are defined with respect to a fixed spectral measure E defined on the Borel set of $\mathbb{R}^k$ for some $k \in \mathbb{N}$ and whose values are projections in the Hilbert space $\mathcal{H}$.

In the quantum logic approach to quantum mechanics it is usually assumed that the propositions associated with a physical system form an orthocomplemented partially ordered set usually possessing some additional structure. Physically a proposition may be interpreted as an observable which only gives the values 0 or 1 whenever it is measured. For further details and a bibliography of this subject we refer the reader to the review article Greechie and Gudder (1973).

In Jauch and Piron's formulation of quantum logic it is possible to deduce the existence of a Hilbert space describing the system from the axioms for the propositional system. The propositions may then be identified with the orthomodular lattice of projections on this Hilbert space. For further details and a precise statement of this result see Piron (1976) or Greechie and Gudder (1973).

We now show that the projections in the *-algebra $\mathcal{A}_\pi$ form an orthomodular lattice satisfying all but one of Piron's axioms for a propositional system (Piron (1976)) namely the completeness property. First we review some definitions.

An orthomodular lattice is a bounded lattice $\mathcal{L}$ (with $\sup\mathcal{L}$ and $\inf\mathcal{L}$ to be denoted by I and 0 respectively) together with a mapping $M \mapsto M^\perp$ from $\mathcal{L}$ to $\mathcal{L}$ such that

[1] $M^{\perp\perp} = M$ for all $M \in \mathcal{L}$,

[2] if $M,N \in \mathcal{L}$ with $M \leq N$ then $N^\perp \leq M^\perp$,

[3] $M \vee M^\perp = I$ for all $M \in \mathcal{L}$,

[4] if $M,N \in \mathcal{L}$ with $M \leq N$ then $M\vee(N\wedge M^\perp) = N$.

(Property [4] is called the orthomodular law.)

Two elements M and N of an orthomodular lattice $\mathcal{L}$ are said to be compatible if the sublattice generated by $\{M,N,M^\perp,N^\perp\}$ is distributive. An orthomodular lattice $\mathcal{L}$ may be called irreducible if the only elements compatible with every element of $\mathcal{L}$ are 0 and I. An element A of an orthomodular lattice $\mathcal{L}$ is called an atom if there are no elements M of $\mathcal{L}$ with $0 < M < A$. $\mathcal{L}$ is said to be atomic if for all $M \in \mathcal{L}$ with $M \neq 0$ there is an atom A with $A \leq M$. Finally an orthomodular lattice $\mathcal{L}$ is said to satisfy the covering law if whenever A is an atom in $\mathcal{L}$ with $M\wedge A = 0$ for all $M \in \mathcal{L}$ then there are no elements N of $\mathcal{L}$ with $M < N < M\vee A$.

Physically $M^{\perp}$ represents the negation of the proposition M. For a discussion of the physical motivation behind the above definitions see Piron (1976). A detailed discussion of the atomicity property and the covering law will be found in Jauch and Piron (1979).

We shall denote the lattice of all projections in the Hilbert space $\mathcal{H}$ by $\mathcal{P}(\mathcal{H})$. Since the projections in $\mathcal{A}_{\mathcal{I}}$ and $\bar{\mathcal{A}}_{\mathcal{I}}$ all belong to $\mathcal{P}(\mathcal{H})$ they inherit a partial ordering from $\mathcal{P}(\mathcal{H})$ and we shall assume that they are ordered in this way throughout this section. Our theorem exhibits some properties of these partially ordered sets of projections.

(14.1) Lemma. E is a compact spectral measure if and only if $I \in \mathcal{A}_L$.

Proof. Let S be the spectrum of E then $E(S) = I$ (see (A6)). If $I \in \mathcal{A}_L$ then $I = E(R)IE(R)$ for some bounded Borel set R. Now $E(\bar{R}) = I$ where $\bar{R}$ is the closure of R so $S \subseteq \bar{R}$ and hence S is bounded and it follows that E is compact. Conversely suppose E is compact then S is bounded so $E(S) \in \mathcal{A}_L$ and hence $I \in \mathcal{A}_L$. ∎

(14.2) Lemma. Let $A \in \mathbb{B}(\mathcal{H})$ then $A \in \bar{\mathcal{A}}_L$ if and only if there is a sequence B_r of bounded Borel sets in $\mathbb{R}^k$ with $E(B_r)AE(B_r) \longrightarrow A$ in the operator norm.

Proof. The proof is identical to an argument used in the proof of (8.7). ∎

(14.3) Theorem. [1] The projections in $\mathcal{A}_I$ are an irreducible atomic orthomodular sublattice of $\mathcal{B}(\mathcal{H})$ in which the covering law holds. If E is not a compact spectral measure then this lattice is not complete.

[2] If E is not a compact spectral measure then the projections in $\bar{\mathcal{A}}_I$ are not a complete lattice.

Proof. It is easily verified that a projection M belongs to $\mathcal{A}_I$ if and only if $M \in \mathcal{A}_L$ or $M^\perp \in \mathcal{A}_L$ (since $\mathcal{A}_I = \mathcal{A}_L + \mathbb{C}I$). A similar property holds for projections in $\bar{\mathcal{A}}_I = \bar{\mathcal{A}}_L + \mathbb{C}I$.

[1] Let M and N be projections in $\mathcal{A}_I$. If $M \in \mathcal{A}_L$ then $M = E(R)ME(R)$ for some bounded Borel set R and now

$$M\wedge N \leq M \leq E(R)$$

which implies $M\wedge N = E(R)(M\wedge N)E(R)$ so $M\wedge N \in \mathcal{A}_L$. Hence if one of M,N belongs to $\mathcal{A}_L$ we have $M\wedge N \in \mathcal{A}_L$ Now suppose neither M nor N belong to $\mathcal{A}_L$ then $M^\perp$ and $N^\perp$ belong to $\mathcal{A}_L$ so there are bounded Borel sets R and S with

$$M^\perp = E(R)M^\perp E(R) \quad \text{and} \quad N^\perp = E(S)N^\perp E(S).$$

This implies $M^\perp \leq E(R)$ and $N^\perp \leq E(S)$ so

$$M^\perp \vee N^\perp = E(R) \vee E(S) = E(R\cup S),$$

giving

$$M^\perp \vee N^\perp = E(R\cup S)(M^\perp \vee N^\perp)E(R\cup S).$$

Since $R\cup S$ is bounded we have $M^\perp \vee N^\perp \in \mathcal{A}_L$ and now by de Morgan's law (Blyth and Janowitz (1972) p162)

$$M \wedge N = I - M^{\perp} \vee N^{\perp} \in \mathcal{A}_{\mathcal{I}} .$$

We have now shown that $M \wedge N$ belongs to $\mathcal{A}_{\mathcal{I}}$ whenever M and N belong to $\mathcal{A}_{\mathcal{I}}$. If M and N belong to $\mathcal{A}_{\mathcal{I}}$

then this property together with de Morgan's law implies that $M \vee N = I - M^{\perp} \wedge N^{\perp}$ also belongs to $\mathcal{A}_{\mathcal{I}}$. Thus the projections in $\mathcal{A}_{\mathcal{I}}$ are a sublattice of $\mathcal{P}(\mathcal{H})$.

Next we show that the projection lattice of $\mathcal{A}_{\mathcal{I}}$ is irreducible i.e. 0 and I are the only projections in $\mathcal{A}_{\mathcal{I}}$ which are compatible with every projection $M \in \mathcal{A}_{\mathcal{I}}$. Since compatibility of projections is equivalent to commutativity (Piron (1976) p42) this follows from the fact that $\mathcal{A}_{\mathcal{I}}' = \mathbb{C}I$ (since $\mathcal{A}_{\mathcal{I}}$ is irreducible by (13.3)).

We now show that the projection lattice of $\mathcal{A}_{\mathcal{I}}$ is atomic. Let M be a non-zero projection in $\mathcal{A}_{\mathcal{I}}$. If $M \in \mathcal{A}_{L}$ let f be a unit vector in the range of M and let P_f be the projection onto the subspace spanned by f. Then

$$P_f \leq M = E(R)ME(R) \leq E(R)$$

for some bounded Borel set R and now P_f is clearly an atom. If $M \notin \mathcal{A}_{L}$ then $M^{\perp} \in \mathcal{A}_{L}$ so $M^{\perp} = E(R)M^{\perp}E(R)$ for some bounded Borel set R. Let S be a bounded Borel set with $R \cap S = \phi$ and $E(S) \neq 0$ (if no such S exists then the spectrum of E must be a subset of R so $\mathcal{A}_{\mathcal{I}} = \mathcal{B}(\mathcal{H})$ by (13.5) and in this case the projection lattice of $\mathcal{A}_{\mathcal{I}}$ is clearly atomic). Now (by considering a unit vector in the range of E(S)) there is a one-dimensional projection A with

$$A \leq E(S) \leq E(R)^{\perp} \leq M .$$

It follows that A is an atom in the projection lattice of $\mathcal{A}_{\mathcal{I}}$ with $A \leq M$. Hence the projection lattice of $\mathcal{A}_{\mathcal{I}}$ is atomic.

It is easily checked that every atom in $\mathcal{A}_{\mathcal{I}}$ is a one-dimensional projection and the covering law in the projection lattice of $\mathcal{A}_{\mathcal{I}}$ now follows from the covering law in $\mathcal{P}(\mathcal{H})$. The orthomodular law in the projection lattice of $\mathcal{A}_I$ follows from the orthomodular law in $\mathcal{P}(\mathcal{H})$.

To complete the proof of [1] we show that the projections in $\mathcal{A}_I$ are not a complete lattice when the spectral measure E is not compact. Suppose that the spectrum Λ of E is not compact then by (13.4) Λ is the union of a disjoint sequence S_r of bounded Borel sets such that for each $r \in \mathbb{N}$, $E(S_r) \neq 0$ and S_r lies outside a closed ball of radius $r-1$ centred at the origin. We shall show that the set $\{E(S_{2r-1}) : r \in \mathbb{N}\}$ has no supremum in the projection lattice of $\mathcal{A}_I$. By way of contradiction suppose that $M \in \mathcal{A}_I$ is a supremum for this set. Note that $E(S_{2r})^{\perp}$ is an upper bound for the above set for every $r \in \mathbb{N}$ so taking an infimum in $\mathcal{P}(\mathcal{H})$ we have

$$M \leq \bigwedge_{r=1}^{\infty} E(S_{2r})^{\perp} = E(S_1 \cup S_3 \cup \ldots).$$

But M is an upper bound for $\{E(S_{2r-1}) : r \in \mathbb{N}\}$ in $\mathcal{P}(\mathcal{H})$ so $M \geq E(S_1 \cup S_3 \cup \ldots)$. Hence $M = E(S_1 \cup S_3 \cup \ldots)$ so $E(S_1 \cup S_3 \cup \ldots)$ belongs to $\mathcal{A}_I$. Now we must have $E(S_1 \cup S_3 \cup \ldots) \in \mathcal{A}_L$ or $E(S_2 \cup S_4 \cup \ldots) = E(S_1 \cup S_3 \cup \ldots)^{\perp} \in \mathcal{A}_L$. Suppose $E(S_1 \cup S_3 \cup \ldots) \in \mathcal{A}_L$ then

$$\begin{aligned} E(S_1 \cup S_3 \cup \ldots) &= E(R)E(S_1 \cup S_3 \cup \ldots)E(R) \\ &= E(R \cap (S_1 \cup S_3 \cup \ldots)) \\ &= E\left(\bigcup_{r=1}^{\infty} R \cap S_{2r-1}\right) \end{aligned}$$

for some bounded Borel set R. Now since R is bounded and each S_r lies outside a ball of radius $r-1$ we have $R \cap S_{2j-1} = \phi$ for some j. Since $E(S_{2j-1}) \neq 0$ we can choose a non-zero vector f in the range of $E(S_{2j-1})$ and now

$$\begin{aligned} f &= E(S_{2j-1})f \\ &= E(S_1 \cup S_3 \cup \ldots)f \\ &= E\left(\bigcup_{r=1}^{\infty} (R \cap S_{2r-1})\right)f, \end{aligned}$$

and since the sequence S_r is disjoint this implies

$$f = E(R_n \cap S_{2j-1})f = 0$$

which is a contradiction since f was chosen to be non-zero. If $E(S_2 \cup S_4 \cup \dots) \in \mathcal{A}_L$ then we arrive at a contradiction in a similar way. Hence the assumption that the set $\{E(S_{2r-1}) : r \in \mathbb{N}\}$ has a supremum in the projection lattice of $\mathcal{A}_L$ leads to a contradiction so this lattice is not complete.

[2] We now show that the projections in $\bar{\mathcal{A}}_I$ do not form a complete lattice. As in the proof of [1] we can find a sequence S_r of disjoint Borel sets in $\mathbb{R}^k$ such that for each r, $E(S_r) \neq 0$, S_r lies outside a ball of radius $r-1$ centred at the origin and the set $\{E(S_{2r-1}) : r \in \mathbb{N}\}$ has no supremum in $\mathcal{A}_I$. By way of contradiction suppose that $M \in \bar{\mathcal{A}}_I$ is a supremum for this set, then as in the proof of [1] we must have $M = E(S_1 \cup S_3 \cup \dots)$ It follows that $E(S_1 \cup S_3 \cup \dots) \in \bar{\mathcal{A}}_L$ or $E(S_2 \cup S_4 \cup \dots) \in \bar{\mathcal{A}}_L$. If $E(S_1 \cup S_3 \cup \dots) \in \bar{\mathcal{A}}_L$ then by (14.2) there is a sequence R_r of bounded Borel sets with

$$E(R_r)E(S_1 \cup S_3 \cup \dots)E(R_r) \longrightarrow E(S_1 \cup S_3 \cup \dots)$$

in the operator norm. Now

$$E(R_r)E(S_1 \cup S_3 \cup \dots)E(R_r) = E(R_r \cap (S_1 \cup S_3 \cup \dots)).$$

It follows from the properties of S_r that this sequence is not eventually constant and so does not converge in the operator norm giving a contradiction. If $E(S_2 \cup S_4 \cup \dots)$ belongs to $\bar{\mathcal{A}}_L$ then we arrive at a contradiction in a similar way. Hence the projections in $\bar{\mathcal{A}}_I$ are not a complete lattice. ∎

15. CONCLUSIONS AND PROSPECTS.

We have shown that an algebraic formulation of quantum mechanics is possible in which the non-locality problems associated with the Einstein, Podolsky and Rosen paradox and the de Broglie paradox do not arise at large distances. At small distances the formulation possesses the usual non-locality property. In our approach once the system reaches "infinity" the only relevant observables form a commutative algebra so the properties of a system at infinity are classical.

This is only an approximation to the situation which arises in practice where quantum mechanical particles may be regarded as widely separated when they are only a few metres apart.

The formulation we have presented is not meant to be a rigid and final structure. There is room for much further development in our approach. A few ideas for future research are listed below.

It would be of interest on the one hand to determine the largest asymptotic algebra on $L^2(\mathbb{R}^n)$ for which spatially separating coherent states become disjoint in an infinite time limit and on the other hand to find the smallest algebra having this property and which also contains all local observables.

We have restricted our attention in this work to particles without spin. For a system with spin one may give similar definitions of spatially separating states and spatially separating systems. In this case the approximate localisation of the system at large time as described in (6.1) and (6.2) is easily generalised. For a particle with spin the Hilbert space may be realised as a tensor product of $L^2(\mathbb{R}^n)$ with a finite dimensional Hilbert space. The result of (6.1) can then be extended by a similar argument to that used in the proof of (7.4). We can then relate spatial separation and disjoint momentum values as in (7.3). Since the only measurements performed in deriving the de Broglie paradox are position measurements the inclusion of spin does not affect our result that spatially separating coherent states become disjoint in an infinite time limit. For a single particle there is no difficulty in admitting spin operators as observables at infinity.

The extension of the present formalism to a two particle system with spin seems to be a much more difficult problem. One difficulty here is related to the fact that (unlike the de Broglie paradox) the Einstein, Podolsky and Rosen paradox may be derived entirely by considering spin measurements performed on distant particles. (Bohm (1951) pp611-623, Selleri and Tarozzi (1980) pp8-11). Initial investigations show that by taking the C*-algebra of the combined system to be the C*-algebra generated by all one particle observables our approach will not lead to an asymptotic resolution of the Einstein, Podolsky and Rosen paradox. It is hoped that an investigation of the precise relationship between one and two particle observables at infinity will enable us to resolve this difficulty. We conjecture that a more general formulation in terms of quantum logic will enable us to give a satisfactory inclusion of spin in our theory.

In conclusion we have presented an axiomatic approach to quantum theory which for many simple systems provides an asymptotic resolution of the paradoxes associated with non-locality. Although we have restricted the set of operators admitted as observables we still have sufficient observables to approximate any of the usual quantum mechanical observables. Finally work is continuing which we hope will extend our approach to more general systems.

APPENDIX.

(A1). SWOPPING OF LIMITS.

Let u_r be a sequence of bounded functions from $\mathbb{R}$ to some metric space M. If the sequence u_r converges uniformly to u and $\lim_{r\to\infty} \lim_{t\to\infty} u_r(t)$ exists then $\lim_{t\to\infty} u(t)$ exists and these two limits are equal.

Proof. Let $L_r = \lim_{t\to\infty} u_r(t)$ and $L = \lim_{r\to\infty} \lim_{t\to\infty} u_r(t)$ then letting d be the metric on M we have $d(L_r, L) \to 0$. Define functions v_r and v from $[0,\infty)$ to M by

$$v_r(t) = \begin{cases} u_r(1/t) & t > 0 \\ L_r & t = 0 \end{cases}$$

and

$$v(t) = \begin{cases} u(1/t) & t > 0 \\ L & t = 0 \end{cases}$$

then each v_r is continuous at 0 and

$$\begin{aligned} \sup_{t\geq 0} d(v_r(t), v(t)) &= \max\left\{ \sup_{t>0} d(u_r(1/t), u(t)),\ d(L_r, L) \right\} \\ &\leq \max\left\{ \sup_{t\in\mathbb{R}} d(u_r(t), u(t)),\ d(L_r, L) \right\} \\ &\to 0 \qquad (r\to\infty) \end{aligned}$$

since u_r converges uniformly to u and $d(L_r, L) \to 0$. Hence v_r converges uniformly to v so v is continuous at 0 (e.g. Sutherland (1975) p121) and the result follows.

(A2). A LEMMA ON STRONG CONVERGENCE.

Let $\mathcal{H}_1$ and $\mathcal{H}_2$ be Hilbert spaces and let A_r and B_r be sequences in $\mathbb{B}(\mathcal{H}_1)$ and $\mathbb{B}(\mathcal{H}_2)$ respectively. If A_r converges strongly to A in $\mathbb{B}(\mathcal{H}_1)$ and B_r converges strongly to B in $\mathbb{B}(\mathcal{H}_2)$ then $A_r \otimes B_r$ converges strongly to $A \otimes B$.

Proof. It follows from the uniform boundedness principle that the sequence A_r is bounded (Weidmann (1980) p75). Suppose A_r converges strongly to 0 in $\mathbb{B}(\mathcal{H}_1)$ then since

$$\|(A_r \otimes I) f \otimes g\| = \|A_r f\| \|g\| \to 0$$

it follows that $(A_r \otimes I) h \to 0$ for all h in some dense linear manifold in $\mathcal{H}_1 \otimes \mathcal{H}_2$. Now let $h \in \mathcal{H}_1 \otimes \mathcal{H}_2$ be arbitrary then there is a sequence h_k converging to h with

$$\lim_{r \to \infty} (A_r \otimes I) h_k = 0 \qquad (\forall k \in \mathbb{N}).$$

Define $u_k(r) = (A_r \otimes I) h_k$ and $u(r) = (A_r \otimes I) h$ then

$$\begin{aligned} \sup_{r \in \mathbb{N}} \| u_k(r) - u(r) \| &= \sup_{r \in \mathbb{N}} \|(A_r \otimes I)(h_k - h)\| \\ &\leq \sup_{r \in \mathbb{N}} \|A_r\| \, \|h_k - h\| \\ &\leq K \, \|h_k - h\| \end{aligned}$$

where K is a bound for the sequence A_r. Hence u_k converges uniformly to u so by (A1) $\lim_{r \to \infty} (A_r \otimes I) h = 0$. Now suppose A_r converges strongly to A then replacing A_r by $A_r - A$ in the above we deduce that $A_r \otimes I \to A \otimes I$ strongly. Similarly $I \otimes B_r$ converges strongly to $I \otimes B_r$ and multiplying gives $A_r \otimes B_r \to A \otimes B$ strongly (Weidmann (1980) p80).

(A3). A THEOREM OF J.D. DOLLARD.

In this appendix we rewrite a theorem of Dollard (Dollard 1969) in a notation which will be useful for some of our applications.

(A3.1) Lemma. For each $a \in \mathbb{R}$ the formula

$$(D_a f)(x) = |a|^{n/2} f(ax) \qquad f \in L^2(\mathbb{R}^n),\ x \in \mathbb{R}^n$$

defines an operator in the Hilbert space $L^2(\mathbb{R}^n)$. If $a \neq 0$ then D_a is unitary and $D_a^{-1} = D_{1/a}$.

Proof. Suppose $a \neq 0$ then the mapping $x \mapsto ax$ is a coordinate transformation on $\mathbb{R}^n$ having Jacobian determinant a^n. It follows from the transformation formula for multiple Lebesgue integrals (Apostol (1974) pp416-421) that if $f \in L^2(\mathbb{R}^n)$ then $D_a f \in L^2(\mathbb{R}^n)$ and $\|f\|^2 = \|D_a f\|^2$. It is now clear that D_a is an isometry and that D_a is invertible with $D_a^{-1} = D_{1/a}$ and it follows that D_a is unitary. ∎

(A3.2) Let F_0 be the unitary operator in $L^2(\mathbb{R}^n)$ defined by

$$(F_0 f)(y) = (2\pi)^{-n/2} \int f(x) e^{-ixy}\, dx$$

for $f \in L^1(\mathbb{R}^n) \cap L^2(\mathbb{R}^n)$ and $y \in \mathbb{R}^n$, and extended by continuity to $L^2(\mathbb{R}^n)$. Also let F denote the Fourier transform in $L^2(\mathbb{R}^n)$ defined in (2.2). Then clearly the operators F and $D_\hbar^{-1} F_0$ coincide on a dense linear manifold and hence are equal.

(A3.3) Lemma. For $t \neq 0$ define an operator C_t in $L^2(\mathbb{R}^n)$ by

$$(C_t f)(x) = \exp\left(\frac{-in\pi t}{4|t|}\right) |2t|^{-n/2} \exp\left(\frac{iQ^2}{4t}\right) (F_0 f)\left(\frac{x}{2t}\right)$$

for all $f \in L^2(\mathbb{R}^n)$ and all $x \in \mathbb{R}^n$ where F_0 is the operator defined in (A3.2). Let U be the free particle evolution group in for a particle of mass m them

$$U_t = C_{t\hbar/2m} \exp\left(\frac{im}{2\hbar t} Q^2\right)$$

and

$$\lim_{t\to\pm\infty} \| U_t f - C_{t\hbar/2m} f \| = 0 \qquad \forall f \in L^2(\mathbb{R}^n).$$

Proof. This is just a rearrangement of a result of Dollard (Dollard (1969)) (and which may be found in the textbook Amrein, Jauch and Sinha (1977)) to include the constants $\hbar$ and m which the latter authors take to be 1 and 1/2 respectively. Define

$$U_t^0 = F_0^{-1} \exp(-itQ^2) F_0$$

then $F_0 f$ is the vector Amrein, Jauch and Sinha denote by $\tilde{f}$ and U_t^0 is the operator these authors denote by U_t (Amrein, Jauch and Sinha (1977) pp42-3, p118). The operator C_t defined in the statement of the lemma coincides with that defined in Amrein, Jauch and Sinha (1977) (p120, p123) so the equalities (3.47) and (3.48) on page 123 of this reference may be written as

$$U_t^0 = C_t \exp\left(\frac{iQ^2}{4t}\right) \qquad (*)$$

$$\lim_{t\to\pm\infty} \| U_t^0 f - C_t f \| = 0 \qquad (**)$$

Now the free particle evolution group for a particle of mass m is given by

$$U_t = \exp\left(\frac{-it}{\hbar} \frac{1}{2m} P^2\right)$$

so

$$\begin{aligned} U_t &= F^{-1} \exp\left(\frac{-it}{2m\hbar} Q^2\right) F \\ &= F_0^{-1} D_\hbar \exp\left(\frac{-it}{2m\hbar} Q^2\right) D_\hbar^{-1} F_0 \\ &= F_0^{-1} \exp\left(\frac{-it}{2m} Q^2\right) F_0 \\ &= U^0_{t\hbar/2m} \end{aligned}$$

and the results stated in the lemma follow on replacing t by $t\hbar/2m$ in $(*)$ and $(**)$.

∎

(A3.4) <u>Corollary</u>. For $t \neq 0$ let C_t° be the operator in $L^2(\mathbb{R}^n)$ given by

$$C_t^\circ = \exp\left(\frac{-in\pi t}{4|t|}\right) D_{m/t} \exp\left(\frac{itQ^2}{2m\hbar}\right) F$$

where F is the Fourier transform and $D_{m/t}$ is the operator defined in (A3.1). If U is the free particle evolution group for a particle of mass m then

$$U_t = C_t^\circ \exp\left(\frac{imQ^2}{2\hbar t}\right)$$

and

$$\lim_{t\to\pm\infty} \| U_t f - C_t^\circ f \| = 0$$

for all f in $L^2(\mathbb{R}^{2n})$.

<u>Proof</u>. By (A3.1) and (A3.2) the operator C_t in (A3.3) may be written as

$$C_t = \exp\left(-\frac{in\pi t}{4|t|}\right) \exp\left(\frac{iQ^2}{4t}\right) D_{2t}^{-1} D_\hbar F$$

$$= \exp\left(\frac{-in\pi t}{4|t|}\right) D_{\hbar/2t} D_{2t/\hbar} \exp\left(\frac{iQ^2}{4t}\right) D_{\hbar/2t} F$$

$$= \exp\left(\frac{-in\pi t}{4|t|}\right) D_{\hbar/2t} \exp\left(\frac{itQ^2}{\hbar^2}\right) F$$

so

$$C_{t\hbar/2m} = \exp\left(\frac{-in\pi t}{4|t|}\right) D_{m/t} \exp\left(\frac{itQ^2}{2m\hbar}\right) F$$

$$= C_t^\circ$$

and the result now follows from (A3.3). ∎

(A4) COMMUTATIVITY.

Let H be a self-adjoint operator in a Hilbert space $\mathcal{H}$ and let $A \in \mathbb{B}(\mathcal{H})$ then these are equivalent

[1] $AH \subseteq HA$;

[2] A commutes with all the spectral projections of H;

[3] A commutes with $u(H)$ for every bounded Borel function $u: \mathbb{R} \to \mathbb{C}$;

[4] A commutes with $\exp(itH)$ for every $t \in \mathbb{R}$.

Outline of Proof. First let $C \in \mathbb{B}(\mathcal{H})$ be unitary. It follows from the spectral theorem for bounded normal operators and Halmos (1957) (theorem 4 on p61, theorem 2 on p65) that $AC = AC$ if and only if A commutes with the spectral projections of C. Now let H be a self-adjoint operator in $\mathcal{H}$.

([1] $\Rightarrow$ [2]) Let $C = (H - iI)(H + iI)^{-1}$ be the Cayley transform of H, where $(H + iI)^{-1} \in \mathbb{B}(\mathcal{H})$ and $\mathrm{dom}(H - iI) = \mathrm{ran}(H + iI)^{-1}$. Assuming [1] a simple calculation shows that $AC = CA$ and since C is unitary we have (from the initial remarks) that A commutes with the spectral projections of C. But the set of all spectral projections of H coincides with the set of all spectral projections of C and [2] follows. For further details of the relationship between C and H see e.g. Rudin (1973) (pp338-340, pp348-349).

The implication $[2] \Rightarrow [3]$ follows from Halmos (1957) (theorem 4 on p61) while $[3] \Rightarrow [4]$ is obvious.

($[4] \Rightarrow [1]$) The domain of H is given by

$$\mathrm{dom}(H) = \{f \in \mathcal{H} : \lim_{t \to 0} \tfrac{1}{t}(\exp(-itH) - I)f \text{ exists}\}$$

and we have

$$Hf = \lim_{t \to 0} \tfrac{i}{t}(\exp(-itH) - I)f$$

for all $f \in \mathrm{dom}(H)$ (Weidmann (1980) pp220-221). It is easily verified that if A commutes with every operator $\exp(-itH)$ then $AHf = HAf$ for all f in $\mathrm{dom}(H)$ so $HA \subseteq AH$.

(A5). STATES ON IRREDUCIBLE OPERATOR ALGEBRAS.

Let $\mathcal{A}$ be an irreducible C*-algebra of operators on a Hilbert space $\mathcal{H}$ and let f be a unit vector in $\mathcal{H}$. Define a state ω on $\mathcal{A}$ by

$$\omega(A) = \langle f|Af\rangle \qquad \text{for all } A \in \mathcal{A}.$$

Let $i: \mathcal{A} \to \mathbb{B}(\mathcal{H})$ be the inclusion defined by $i(A) = A$ for all $A \in \mathcal{A}$ then

[1] $(\mathcal{H}, i)$ is unitarily equivalent to the canonical cyclic representation associated with ω,

[2] ω is pure,

[3] if $g \in \mathcal{H}$ is such that $\omega(A) = \langle g|Ag\rangle$ for all $A \in \mathcal{A}$ then $g = af$ for some $a \in \mathbb{C}$.

<u>Proof</u> [1]: Since $\mathcal{A}$ is irreducible f is a cyclic vector for $\mathcal{A}$ and the result follows from the uniqueness property of the canonical cyclic representation (Bratteli and Robinson (1979) p47,p56).

[2]: By [1] the canonical cyclic representation associated with ω is irreducible so ω is pure (Bratteli and Robinson (1979) p57).

[3]: see Dixmier (1977) p45.

(A6). THE SPECTRUM OF A SPECTRAL MEASURE.

Our definitions are all taken from Halmos (1957) sections 38 and 39. The resolvent of a spectral measure E defined on the Borel sets of $\mathbb{R}^k$ is defined to be the union of all open sets U for which $E(U) = 0$. The spectrum of E is defined to be the complement of the resolvent of E in $\mathbb{R}^k$. E is said to be compact if its spectrum is compact.

<u>Theorem</u>. Let E be a spectral measure on the Borel sets of $\mathbb{R}^k$ and let S be the spectrum of E then

[1] S is the intersection of all closed sets C for which $E(C) = I$,

[2] E is compact if and only if S is bounded,

[3] $E(S) = I$.

<u>Proof</u>. [1] follows from one of de Morgan's laws and [2] is a consequence of the Heine-Borel theorem and its converse (for subspaces of $\mathbb{R}^k$) since S is clearly closed.

Since every measure on the Borel sets of $\mathbb{R}^k$ is regular (Halmos (1950) 51 p220,228) it follows that E is regular (Halmos (1957) p63) and [3] now follows from Halmos (1957) (theorem 1 on p62).

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